MAT 127 — Calculus C: Course Notes

Part II - Differential Equations

These notes are based on the textbook *Calculus: Concepts and Contexts, 5th Edition, by James Stewart and Steve Kokoska*, on which the course is based. They provide a summary of the material covered during the class sessions. Students are strongly encouraged to supplement these notes by studying the textbook and practicing with the exercises provided there. It is also essential to attend office hours in order to ask questions and clarify any difficulties. Simply reading these notes is not sufficient, especially given the limited amount of time available during class. These notes will be updated after each week of classes.

Contents

1 Differential Equations **2**

2 Examples of Some Famous Models **4**

3 Solutions of Differential Equations **10**

4 Initial-Value Problems **16**

5 Slope Fields and Euler's Method **19**

1 Differential Equations

What Is a Differential Equation?

Definition. Differential Equation

A *differential equation* is an equation involving an unknown function y and one or more of its derivatives: y', y'', y''', $\cdots$

The way we write differential equations may vary a lot depending on the situation. For example, while the notations y' or $\frac{dy}{dx}$ are commonly used in mathematics, the notation $\dot{y}$ may be encountered in physics. Moreover, the choice of notation for the function and its variable may also vary. Here are some examples:

1. In a differential equation involving y and its derivatives with respect to x, it is understood that y is an unknown function of x. For example, in
$$y' = xy,$$
the unknown is a function $y = y(x)$ and x is the variable.

2. In a differential equation involving f and $\frac{df}{dt}$, the quantity f is regarded as an unknown function of t. For example, in
$$\frac{df}{dt} = f^2 - f,$$
the unknown is a function $f = f(t)$ and t is the variable.

3. In physics, one may write
$$\dot{v} = t - v,$$
where the unknown is a function $v = v(t)$ and t is the variable. Here $\dot{v}$ denotes the derivative of v with respect to time. In mathematical notation, the same differential equation can be written as
$$\frac{dv}{dt} = t - v.$$

Definition. Order of a Differential Equation

The *order* of a differential equation is the order of the highest derivative that appears in the equation. In particular, a differential equation is called a *first-order differential equation* if the highest derivative appearing in the equation is a first derivative, a *second-order differential equation* if the highest derivative appearing in the equation is a second derivative, and so on.

For example,
$$y' = xy$$
is a first-order differential equation, while
$$m\frac{d^2x}{dt^2} = -kx$$
is a second-order differential equation.

Ordinary and Partial Differential Equations

Definition. Ordinary and Partial Differential Equations

An *ordinary differential equation* (ODE) involves an unknown function of a single independent variable and its ordinary derivatives. A *partial differential equation* (PDE) involves an unknown function of several independent variables and its partial derivatives.

For instance, the equation

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y^2 = 5\sin x$$

is a second-order ODE for the unknown function $y = y(x)$. In contrast, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a second-order PDE for an unknown function $u = u(x, y)$. In this course, we will focus on ordinary differential equations.

2 Examples of Some Famous Models

A major source of differential equations comes from physics and from the mathematical modeling of concrete problems. In many situations, the evolution of a quantity over time is described by a differential equation relating the quantity itself to one or more of its derivatives. We now present a few classical examples showing how differential equations arise naturally in applications.

■ Models of Population Growth

Suppose that $P(t)$ denotes the number of individuals in a population at time t. Under ideal conditions, it is reasonable to assume that the rate of growth of the population is proportional to the size of the population.

Since the rate of growth is $\frac{dP}{dt}$, this assumption gives

$$\boxed{\frac{dP}{dt} = kP},$$

where k is a constant of proportionality.

If $k > 0$ and $P(t) > 0$, then

$$\frac{dP}{dt} > 0,$$

so the population is increasing. Moreover, as $P(t)$ becomes larger, the growth rate $\frac{dP}{dt} = kP(t)$ also becomes larger.

A function of the form

$$P(t) = Ce^{kt}$$

is a solution of this differential equation because

$$P'(t) = Cke^{kt} = kCe^{kt} = kP(t).$$

Therefore, the equation $\frac{dP}{dt} = kP$ has a family of solutions

$$\boxed{P(t) = Ce^{kt}},$$

where C is a constant.

If $t = 0$, then

$$P(0) = Ce^{k\cdot 0} = C.$$

Thus, in a population model, the constant C represents the initial population.

Example 1 Bacterial Growth and Doubling Time

The number of bacteria in a culture grows at a rate proportional to the number present. Suppose there are initially 500 bacteria and the population increases to 2000 after 2 hours. Determine the number of bacteria after 12 hours and the doubling time.

Solution

The Malthusian model is $P'(t) = kP(t)$, so $P(t) = Ce^{kt}$. The initial condition $P(0) = 500$ gives $C = 500$. Since $P(2) = 2000$,

$$2000 = 500e^{2k} \quad \Longrightarrow \quad e^{2k} = 4 \quad \Longrightarrow \quad k = \ln 2.$$

Therefore,

$$\boxed{P(t) = 500e^{t\ln 2} = 500 \cdot 2^t.}$$

After 12 hours, the model predicts

$$\boxed{P(12) = 500 \cdot 2^{12} = 2,048,000 \text{ bacteria.}}$$

The *doubling time* t_d is the time required for $P(t_d) = 2P(0)$. In this example,

$$500 \cdot 2^{t_d} = 1000 \qquad \Longrightarrow \qquad \boxed{t_d = 1 \text{ hour.}}$$

More generally, for the model $P(t) = P_0 e^{kt}$ with $k > 0$, the doubling time is $t_d = \frac{\ln 2}{k}$.

■ The Logistic Model

The model

$$\frac{dP}{dt} = kP$$

assumes unlimited growth and is therefore appropriate only under ideal conditions. In reality, an environment has limited resources, so a population cannot usually increase exponentially forever.

Let $M > 0$ denote the *carrying capacity* of the environment. A more realistic model should satisfy the following properties:

1. When P is small compared with M, the growth rate should be approximately proportional to P:

$$\frac{dP}{dt} \approx kP.$$

2. If $P > M$, the population should decrease: $\frac{dP}{dt} < 0$.

Both properties are incorporated into the equation

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}.$$

This equation is called the *logistic differential equation*.

Let us examine its behavior:

1. If P is small compared with M, then $\frac{P}{M}$ is close to 0, so

$$1 - \frac{P}{M} \approx 1,$$

and therefore

$$\frac{dP}{dt} \approx kP.$$

2. If $P = 0$ or $P = M$, then

$$kP\left(1 - \frac{P}{M}\right) = 0,$$

so $P(t) = 0$ and $P(t) = M$ are constant solutions.

3. If $0 < P < M$, then $1 - \frac{P}{M} > 0$, so, assuming $k > 0$, we have $\frac{dP}{dt} > 0$. Thus, the population increases.

4. If $P > M$, then $1 - \frac{P}{M} < 0$, so $\frac{dP}{dt} < 0$. Thus, the population decreases.

Definition. Equilibrium Solution

A constant solution of a differential equation is called an *equilibrium solution.*

For the logistic equation, $P = 0$ and $P = M$ are equilibrium solutions. Moreover, solutions with $0 < P < M$ increase toward M, while solutions with $P > M$ decrease toward M. Thus, the carrying capacity $P = M$ describes the long-term population level predicted by the model.

Derivation and Solution of the Logistic Model

One way to motivate the logistic equation is to consider a population in a closed environment. Let $B(t)$ and $D(t)$ denote its per-capita birth and death rates. The population then satisfies

$$\frac{dP}{dt} = (B(t) - D(t))P.$$

Suppose that the birth rate is constant, $B(t) = B_0 > 0$, and that the death rate increases proportionally to the population, $D(t) = D_0P(t)$, where $D_0 > 0$. We obtain

$$\frac{dP}{dt} = (B_0 - D_0P)P = B_0P\left(1 - \frac{P}{M}\right), \qquad M = \frac{B_0}{D_0}.$$

Thus, this assumption leads to the logistic equation with $k = B_0$ and carrying capacity $M = B_0/D_0$.

We can also find an explicit formula for the solution with initial population $P(0) = P_0 > 0$. For a nonconstant solution, with $P \neq 0, M$, separate the variables:

$$\frac{M}{P(M-P)}\,dP = k\,dt.$$

Using the partial fraction decomposition

$$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P},$$

we obtain

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt,$$

so

$$\ln|P| - \ln|M-P| = kt + C,$$

or equivalently,

$$\ln\left|\frac{P}{M-P}\right| = kt + C.$$

On an interval of definition, the quotient $P/(M-P)$ has a constant sign. Exponentiating and incorporating this sign into a nonzero constant A gives

$$\frac{P}{M-P} = Ae^{kt}.$$

Solving for P,

$$\boxed{P(t) = \frac{MAe^{kt}}{1+Ae^{kt}} = \frac{M}{1+Be^{-kt}},}$$

where $B = 1/A$. For $P(0) = P_0 \neq M$, we find $B = (M-P_0)/P_0$. The same final formula also gives the equilibrium solution $P(t) = M$ when $P_0 = M$. Therefore,

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kt}}, \qquad P_0 > 0,}$$

on intervals where the denominator does not vanish. In particular, for $k > 0$ and $P_0 > 0$, this solution is defined for all $t \geq 0$. The constant solution $P(t) = 0$ must be considered separately if $P_0 = 0$.

If $0 < P_0 \ll M$ and t is not too large, the denominator is approximately Me^{-kt}, so $P(t) \approx P_0 e^{kt}$. Thus, the Malthusian model approximates the initial growth of a small population.

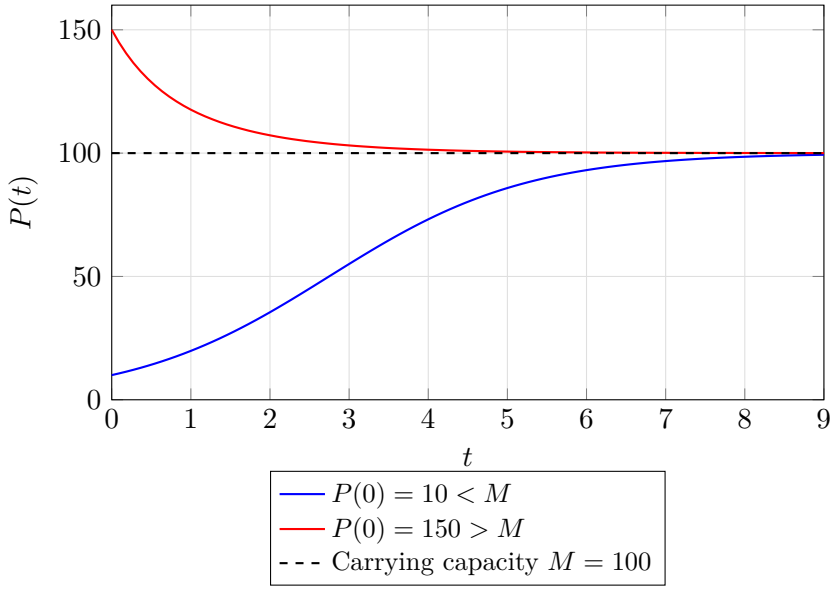


Figure 1: Typical logistic solutions: populations below the carrying capacity increase toward M, while populations above it decrease toward M.

Example 2 Estimating a City's Population with the Logistic Model

The population of a city, measured in thousands, is initially 20. After 10 years, it has increased to 50.87, and after 15 years, to 78.68. Use the logistic model to predict the population after 30 years.

Solution

Write the logistic solution in terms of the initial population $P_0 = 20$:

$$P(t) = \frac{20M}{20 + (M - 20)e^{-kt}}.$$

The observations $P(10) = 50.87$ and $P(15) = 78.68$ yield the system

$$\begin{cases} 50.87 = \dfrac{20M}{20 + (M - 20)e^{-10k}}, \\ 78.68 = \dfrac{20M}{20 + (M - 20)e^{-15k}}. \end{cases}$$

Solving this nonlinear system numerically gives $M \approx 500.37$ and $k \approx 0.1000$. Hence,

$$P(t) \approx \frac{20(500.37)}{20 + (500.37 - 20)e^{-0.1t}}.$$

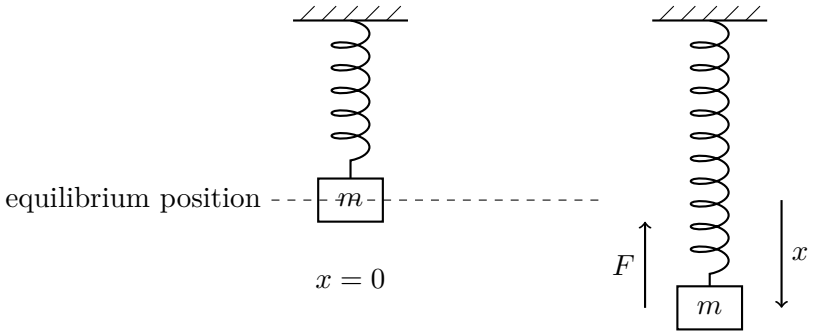
Evaluating at $t = 30$ gives

$$\boxed{P(30) \approx 228 \text{ thousand inhabitants.}}$$

The numerical prediction is approximate because the population measurements have been rounded.

A Model for the Motion of a Spring

Consider an object of mass m attached to a vertical spring. Let $x(t)$ denote the displacement of the object from its equilibrium position.



Hooke's Law states that the restoring force exerted by the spring is proportional to the displacement and acts in the opposite direction:

$$F = -kx,$$

where $k > 0$ is the spring constant. The minus sign indicates that the force always points toward the equilibrium position.

If we ignore external resisting forces such as air resistance and friction, Newton's Second Law gives

$$F = m\frac{d^2x}{dt^2}.$$

Therefore,

$$m\frac{d^2x}{dt^2} = -kx,$$

or equivalently,

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}.$$

This is a second-order differential equation. It says that the acceleration of the object is proportional to its displacement from equilibrium and has the opposite sign.

Oscillations of a Spring

Recall the spring model $m\frac{d^2x}{dt^2} = -kx$, where $m, k > 0$. Setting $\omega = \sqrt{k/m}$, we can rewrite the equation as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

The function $x(t) = A\cos(\omega t - \varphi)$ is a solution, where A and φ are constants. Indeed,

$$x'(t) = -A\omega\sin(\omega t - \varphi), \qquad x''(t) = -A\omega^2\cos(\omega t - \varphi),$$

and therefore

$$x''(t) + \omega^2 x(t) = -A\omega^2\cos(\omega t - \varphi) + \omega^2 A\cos(\omega t - \varphi) = 0.$$

The model predicts an oscillatory motion about the equilibrium position.

A Falling Object with Air Resistance

As another application of Newton's Second Law, consider an object of mass $m > 0$ falling from rest near the Earth's surface. Let $y(t)$ denote the distance traveled downward and $v(t) = y'(t)$ its velocity. Choose the downward direction as positive and assume that the air-resistance force is proportional to the velocity with proportionality constant $k > 0$.

The gravitational force is mg and the resisting force is $-kv$. Newton's Second Law gives

$$m\frac{dv}{dt} = mg - kv, \qquad v(0) = 0.$$

Rewriting this differential equation as

$$\frac{dv}{mg - kv} = \frac{dt}{m}$$

and integrating gives

$$-\frac{1}{k}\ln|mg - kv| = \frac{t}{m} + C.$$

Solving for v and using $v(0) = 0$, we obtain

$$\boxed{v(t) = \frac{mg}{k}\left(1 - e^{-(k/m)t}\right).}$$

Consequently,

$$\boxed{\lim_{t\to+\infty} v(t) = \frac{mg}{k}.}$$

This limit is called the *terminal velocity*: as the object falls, the air resistance eventually balances the gravitational force.

Since $y'(t) = v(t)$ and $y(0) = 0$, integration also gives the displacement

$$\boxed{y(t) = \frac{mg}{k}\left(t - \frac{m}{k}\left(1 - e^{-(k/m)t}\right)\right).}$$

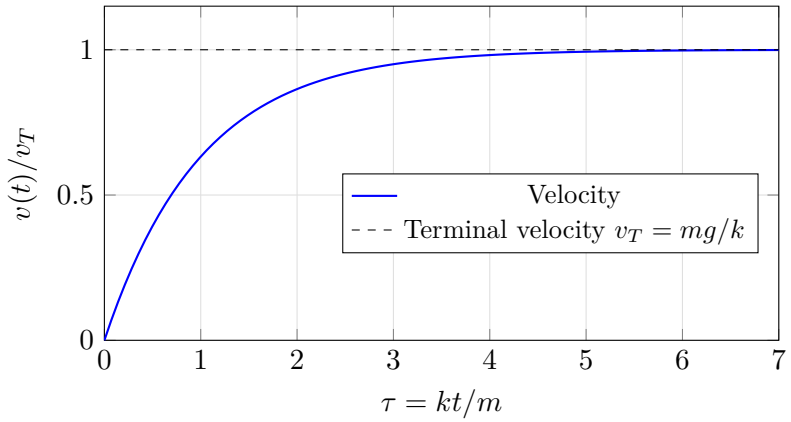


Figure 2: The falling object's velocity approaches the terminal velocity $v_T = mg/k$. The plot uses the dimensionless time $\tau = kt/m$.

3 Solutions of Differential Equations

Definition. Solution of a Differential Equation

A function f defined on an interval I is a *solution* of a differential equation on I if it has the required derivatives there and the equation is satisfied at every point of I when the unknown function is replaced by f and its derivatives by the corresponding derivatives of f.

For example, consider

$$y' = x^3.$$

Since

$$\frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3,$$

every function of the form

$$\boxed{y = \frac{x^4}{4} + C}$$

is a solution.

Definition. General Solution

The *general solution* of a differential equation is a family of functions that describes all the solutions of the equation, usually depending on one or more arbitrary constants.

For the differential equation

$$y' = x^3,$$

the general solution is therefore

$$\boxed{y = \frac{x^4}{4} + C},$$

where $C \in \mathbb{R}$ is an arbitrary constant.

In general, finding all solutions of a differential equation can be difficult. In many situations, we may instead study qualitative properties of the solutions or approximate them numerically. However, we should first review some basic examples for which the general solution can be computed explicitly.

Example 1 Verifying a Solution of a Second-Order Equation

For arbitrary constants $C_1, C_2 \in \mathbb{R}$, consider

$$y(x) = C_1 e^{2x} + C_2 e^{-2x}.$$

We have

$$y'(x) = 2C_1 e^{2x} - 2C_2 e^{-2x}, \qquad y''(x) = 4C_1 e^{2x} + 4C_2 e^{-2x} = 4y(x).$$

Thus, $y'' - 4y = 0$ for every real x. In particular, each choice of C_1, C_2 gives a solution on $\mathbb{R}$.

■ Equations That Can Be Solved by Direct Integration

When the differential equation has the form $y' = f(x)$ with f continuous, every solution is an antiderivative of f. If F is one antiderivative of f, the general solution is

$$\boxed{y(x) = F(x) + C, \qquad C \in \mathbb{R}.}$$

For example, for $y' = \sin x$, the general solution is $y(x) = -\cos x + C$.

■ Exponential and Constant Functions

Two basic examples of differential equations that we should know are:

Theorem. Exponential and Constant Functions

1. The set of solutions of $y' = 0$ is given by the constant functions

$$y(x) = C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

2. The set of solutions of $y' = y$ is given by the exponential functions

$$y(x) = Ce^x,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

We will assume this theorem and use it as a basic building block for solving more complicated differential equations.

■ Method of Separation of Variables

The idea of the *method of separation of variables* is to rewrite the differential equation so that all the terms involving the unknown function are on one side of the equation and all the terms involving the variable are on the other side. We can then integrate both sides.

Definition. Separable Differential Equation

A first-order differential equation is called *separable* if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x).$$

Integrating both sides gives the implicit relation

$$\boxed{\int p(y)\,dy = \int q(x)\,dx + C,}$$

where C is a constant. For an equation of the form $y' = f(x)g(y)$, one may divide by $g(y)$ on intervals where $g(y) \neq 0$; constant solutions satisfying $g(y) = 0$ must be checked separately.

Example 1 Separation of Variables

Consider the differential equation

$$y' = \frac{1}{2}(y^2 - 1).$$

Step 1. Find the constant solutions.

First, observe that the constant functions

$$y = 1 \qquad \text{and} \qquad y = -1$$

are solutions.

Step 2. Separate the variables.

We now look for the nonconstant solutions. Since $y \neq \pm 1$, we can divide by $y^2 - 1$ and write

$$\frac{dy}{y^2 - 1} = \frac{1}{2}\,dt.$$

Step 3. Integrate both sides.

We obtain

$$\int \frac{dy}{y^2-1} = \int \frac{1}{2}\, dt.$$

Using the partial fraction decomposition

$$\frac{1}{y^2-1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right),$$

we obtain

$$\frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = \frac{1}{2}t + C.$$

Multiplying by 2 gives

$$\ln\left|\frac{y-1}{y+1}\right| = t + C.$$

Step 4. Solve for y.

Exponentiating, we obtain

$$\left|\frac{y-1}{y+1}\right| = Ae^t,$$

where $A > 0$ is a constant. Since $Ae^t > 0$ for every t, the quotient $\frac{y-1}{y+1}$ never vanishes. Therefore, on the interval under consideration, it has a constant sign. Hence, either

$$\frac{y-1}{y+1} = Ae^t$$

or

$$\frac{y-1}{y+1} = -Ae^t.$$

By writing $c = \pm A$, where c is a nonzero real constant, we obtain

$$\frac{y-1}{y+1} = ce^t.$$

Solving for y,

$$\begin{aligned} y - 1 &= ce^t(y+1) \\ y - ce^t y &= 1 + ce^t \\ y(1-ce^t) &= 1 + ce^t, \end{aligned}$$

and therefore

$$y = \frac{1+ce^t}{1-ce^t}.$$

Thus, all nonconstant solutions are of the form

$$\boxed{y = \frac{1+ce^t}{1-ce^t}},$$

on any interval on which $1 - ce^t \neq 0$. The choice $c = 0$ also gives the constant solution $y = 1$, while the constant solution $y = -1$ must be added separately.

Hence, all the solutions of the differential equation are

$$\boxed{y = -1 \qquad \text{or} \qquad y = \frac{1+ce^t}{1-ce^t}, \quad c \in \mathbb{R},}$$

on any interval on which $1 - ce^t \neq 0$. The choice $c = 0$ gives the constant solution $y = 1$.

Example 2 Another Example of Separation of Variables

Consider the differential equation

$$y' = xy.$$

Step 1. Find the constant solutions.

The constant function

$$y = 0$$

is a solution.

Step 2. Separate the variables.

We now look for solutions such that $y \neq 0$. We can write

$$\frac{dy}{dx} = xy,$$

and therefore

$$\frac{dy}{y} = x\,dx.$$

Step 3. Integrate both sides.

We obtain

$$\int \frac{dy}{y} = \int x\,dx,$$

so

$$\ln|y| = \frac{x^2}{2} + C.$$

Step 4. Solve for y.

Exponentiating, we obtain

$$|y| = Ae^{x^2/2},$$

where $A > 0$ is a constant. Since $Ae^{x^2/2} > 0$ for every x, the function y never vanishes. Therefore, on the interval under consideration, y has a constant sign. Hence, either

$$y = Ae^{x^2/2}$$

or

$$y = -Ae^{x^2/2}.$$

By writing $C = \pm A$, where C is a nonzero real constant, we obtain

$$y = Ce^{x^2/2}.$$

The constant solution $y = 0$ is obtained by allowing $C = 0$. Therefore, the general solution is

$$\boxed{y = Ce^{x^2/2}, \qquad C \in \mathbb{R}.}$$

Example 3 A Separable Equation with an Implicit Solution

Solve the equation

$$(e^y + 4y^2)\frac{dy}{dx} = 9xe^{3x}.$$

Solution

The variables are already separated. Integrating both sides,

$$\int (e^y + 4y^2)\,dy = \int 9xe^{3x}\,dx.$$

We use integration by parts on the right-hand side:

$$\int 9xe^{3x}\,dx = 3xe^{3x} - \int 3e^{3x}\,dx = e^{3x}(3x-1) + C.$$

Therefore,

$$\boxed{e^y + \frac{4}{3}y^3 = e^{3x}(3x-1) + C.}$$

This is an *implicit solution*: it gives a relation between x and y, even though it may not be possible to solve explicitly for y using elementary functions.

Example 4 Remembering the Constant Solution

Find all solutions of

$$\frac{dy}{dx} = -2xy^2.$$

Solution

First, $y = 0$ is a constant solution. When $y \neq 0$, separation of variables gives

$$\frac{dy}{y^2} = -2x\,dx.$$

Integrating,

$$-\frac{1}{y} = -x^2 + C,$$

so

$$y(x) = \frac{1}{x^2 - C}.$$

Thus, on intervals where the denominator does not vanish, the solutions are

$$\boxed{y = 0 \qquad \text{or} \qquad y = \frac{1}{x^2 - C}, \quad C \in \mathbb{R}.}$$

The constant solution $y = 0$ cannot be obtained by any choice of C in the second formula because we divided by y^2.

First-Order Linear Differential Equations

Definition. First-Order Linear Differential Equation

An equation of the form

$$a(x)y' + b(x)y = r(x),$$

where the coefficients are defined on an interval I and $a(x) \neq 0$, is called a *first-order linear differential equation*. Dividing by $a(x)$ gives the standard form

$$\boxed{y' + p(x)y = q(x),}$$

where $p(x) = b(x)/a(x)$ and $q(x) = r(x)/a(x)$.

The *integrating factor method* solves this equation by multiplying both sides by a suitable function. Choose

$$\mu(x) = e^{\int p(x)\,dx}.$$

Then $\mu'(x) = p(x)\mu(x)$, so the product rule gives

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)y'(x) + \mu'(x)y(x) = \mu(x)(y'(x) + p(x)y(x)).$$

Multiplying the equation by μ therefore yields

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)q(x).$$

Integrating both sides,

$$\mu(x)y(x) = \int \mu(x)q(x)\,dx + C.$$

Thus, the general solution can be written as

$$\boxed{y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)\,dx + C\right), \qquad \mu(x) = e^{\int p(x)\,dx}.}$$

Example 5 Using an Integrating Factor

Solve the initial-value problem

$$\begin{cases} y' + xy = xe^{x^2/2}, \\ y(0) = 1. \end{cases}$$

Solution

Here $p(x) = x$, so an integrating factor is

$$\mu(x) = e^{\int x\,dx} = e^{x^2/2}.$$

Multiplying the equation by this factor and using the product rule gives

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2}.$$

Integrating,

$$e^{x^2/2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C.$$

Therefore,

$$y(x) = \frac{1}{2}e^{x^2/2} + Ce^{-x^2/2}.$$

The initial condition $y(0) = 1$ gives $1 = \frac{1}{2} + C$, so $C = \frac{1}{2}$. Hence,

$$\boxed{y(x) = \frac{1}{2}\left(e^{x^2/2} + e^{-x^2/2}\right) = \cosh(x^2/2).}$$

4 Initial-Value Problems

When solving a differential equation, we are often interested not in the entire family of solutions but in one particular solution that satisfies an additional condition.

A condition of the form

$$y(t_0) = y_0$$

is called an *initial condition*.

Definition. Initial-Value Problem

An *initial-value problem* consists of a differential equation together with one or more initial conditions. The goal is to find a solution of the differential equation that satisfies the given initial conditions.

Geometrically, imposing the initial condition $y(t_0) = y_0$ means selecting from the family of solution curves the curve that passes through the point (t_0, y_0).

For example, consider the initial-value problem

$$\begin{cases} y' = \frac{1}{2}(y^2 - 1), \\ y(0) = 2. \end{cases}$$

We already know that all the solutions of the differential equation are

$$y = -1 \qquad \text{or} \qquad y = \frac{1 + ce^t}{1 - ce^t}.$$

The constant solution $y = -1$ does not satisfy the initial condition $y(0) = 2$, so we only need to consider

$$y = \frac{1 + ce^t}{1 - ce^t}.$$

Using the initial condition $y(0) = 2$, we obtain

$$2 = \frac{1 + c}{1 - c}.$$

Therefore,

$$2 - 2c = 1 + c,$$

so

$$c = \frac{1}{3}.$$

Substituting this value of c into the general solution gives

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}.$$

Thus, the unique solution of the initial-value problem

$$\begin{cases} y' = \frac{1}{2}(y^2 - 1), \\ y(0) = 2 \end{cases}$$

on the maximal interval containing 0, namely $]-\infty, \ln 3[$, is

$$\boxed{y(t) = \frac{3 + e^t}{3 - e^t}}.$$

■ Initial-Value Problems and Direct Integration

For a continuous function f, an initial-value problem of the form

$$\begin{cases} y' = f(x), \\ y(x_0) = y_0 \end{cases}$$

can be solved by integration:

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt.}$$

Indeed, the Fundamental Theorem of Calculus gives $y'(x) = f(x)$, and substituting $x = x_0$ gives $y(x_0) = y_0$.

■ Existence and Uniqueness of Solutions

An initial-value problem does not automatically have a solution, and a solution need not be defined for all real values of the independent variable.

Theorem. Existence and Uniqueness for a First-Order Equation

Consider the initial-value problem

$$\begin{cases} y' = F(x, y), \\ y(x_0) = y_0. \end{cases}$$

If F and its partial derivative $\frac{\partial F}{\partial y}$ are continuous on an open rectangle containing (x_0, y_0), then there exists an interval containing x_0 on which the initial-value problem has a unique solution.

The theorem guarantees existence and uniqueness *locally*, near x_0. It does not imply that the solution can be extended to every $x \in \mathbb{R}$.

For instance, the differential equation $y' = y^2$ has the constant solution $y = 0$ and, for each constant C, the nonconstant solutions

$$y(x) = \frac{1}{C - x}$$

on intervals that do not contain $x = C$. Even though $F(x, y) = y^2$ and $\frac{\partial F}{\partial y} = 2y$ are continuous everywhere, these nonconstant solutions blow up at a finite value of x.

It is also possible that an equation has no real solution at all: for example, $(y')^2 + y^2 = -1$ is impossible because its left-hand side is nonnegative. Conversely, $(y')^2 + y^2 = 0$ has only the solution $y = 0$.

■ The Interval of Definition Matters

Consider the equation

$$\frac{dy}{dx} = \frac{y - 1}{2\sqrt{x}}.$$

The right-hand side is defined for $x > 0$, so we consider solutions on intervals contained in $]0, +\infty[$. Separation of variables gives

$$\frac{dy}{y - 1} = \frac{dx}{2\sqrt{x}},$$

which leads to

$$\ln|y - 1| = \sqrt{x} + C.$$

Thus, the solutions have the form

$$\boxed{y(x) = 1 + Ce^{\sqrt{x}}, \qquad x > 0, \quad C \in \mathbb{R}.}$$

The choice $C = 0$ includes the constant solution $y = 1$. The formula illustrates why the domain of the differential equation must be specified when discussing its solutions.

5 Slope Fields and Euler's Method

■ Slope Fields

In general, it is not always possible to find an explicit formula for the solution of a differential equation. However, a differential equation itself gives information about the slope of its solution curves.

Consider a first-order differential equation of the form

$$y' = F(x, y).$$

If a solution curve passes through a point (x, y), then its slope at that point is

$$F(x, y).$$

For example, consider

$$y' = x + y.$$

At the point $(0, 1)$ the slope of any solution curve passing through this point is

$$y' = 0 + 1 = 1.$$

Thus, near $(0, 1)$ the solution curve looks approximately like a line segment of slope 1.

More generally, we can draw a short line segment with slope $F(x, y)$ at many points (x, y) in the plane.

Definition. Slope Field

For a differential equation

$$y' = F(x, y),$$

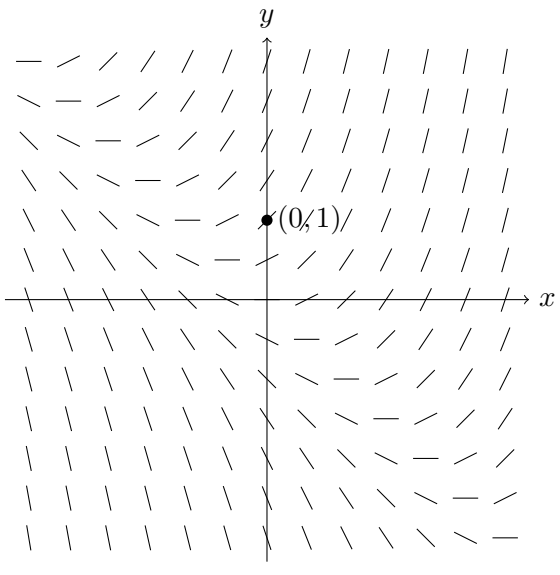
a *slope field*, or *direction field*, is obtained by drawing a short line segment of slope $F(x, y)$ at each of several points (x, y) in the plane.

A slope field gives a graphical picture of the possible solution curves of the differential equation. A solution curve must be tangent to the line segments of the slope field at every point through which it passes.

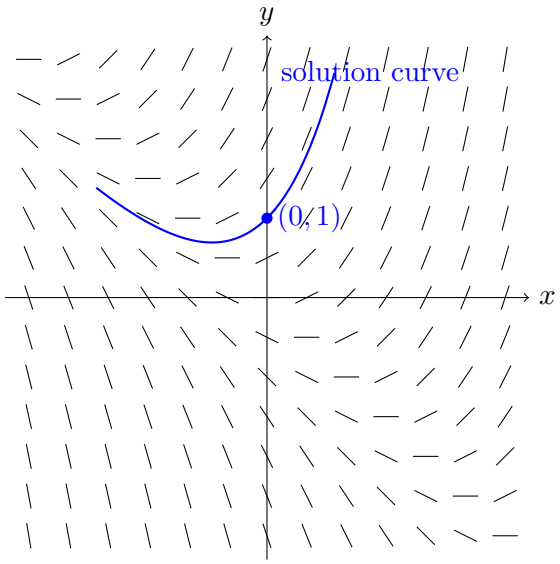
For the differential equation

$$y' = x + y,$$

the following figure shows a portion of the corresponding slope field.



Starting from the point $(0, 1)$ and following the directions indicated by the slope field allows us to sketch the solution curve satisfying the initial condition $y(0) = 1$.



Implicit Solution Curves

A differential equation may also have solutions described by an equation that does not explicitly solve for y. For example, differentiating

$$x^2 + y^2 = 4$$

with respect to x gives

$$2x + 2yy' = 0, \qquad \text{so} \qquad y' = -\frac{x}{y}$$

where $y \neq 0$. The two branches $y = \pm\sqrt{4 - x^2}$, defined for $-2 < x < 2$, describe the upper and lower semicircles. Each branch is a solution of this differential equation on that interval. Together, their graphs lie on the circle centered at the origin with radius 2.

More generally, for every $C > 0$, the equation $x^2 + y^2 = C$ describes a circle centered at the origin with radius $\sqrt{C}$. Its upper and lower semicircles are solution curves of the same differential equation.

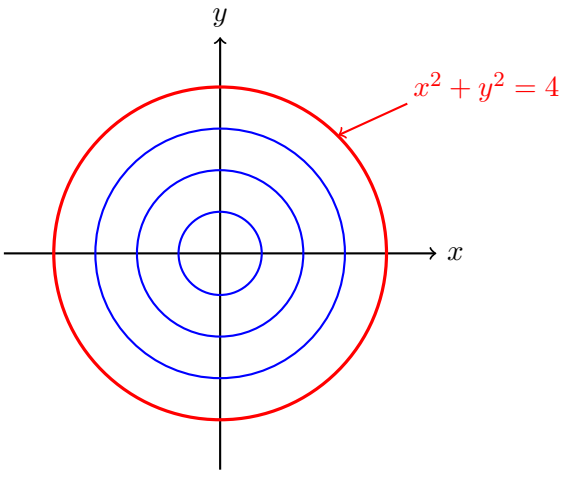


Figure 3: The circles $x^2 + y^2 = C$, for $C > 0$, represent implicit solutions of the differential equation $y' = -x/y$. The red circle corresponds to $C = 4$. Each semicircle, away from $y = 0$, is a solution curve.

What can we learn from the slope field?

The interesting point is that we can recover the shape of these solution curves without explicitly solving the differential equation.

Indeed, consider the equation

$$y' = -\frac{x}{y}.$$

At every point (x, y) with $y \neq 0$, the slope of a solution curve is $-\frac{x}{y}$. We can therefore construct the corresponding slope field.

By following the directions indicated by the small line segments, we can guess the shape of the solution curves.

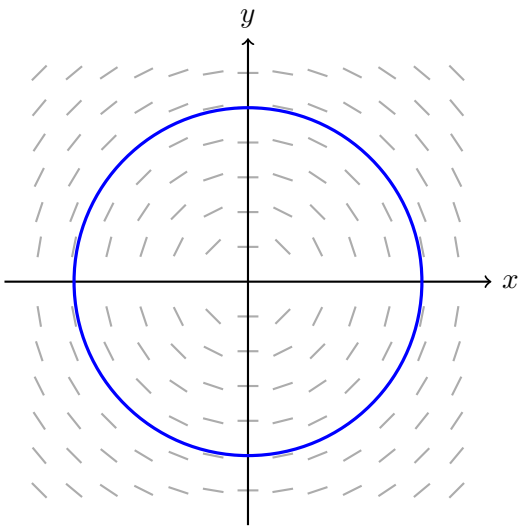


Figure 4: Slope field of $y' = -x/y$. The blue circle illustrates the circular shape suggested by the slope field.

We observe that the directions indicated by the slope field follow a circular pattern. In particular, the blue circle looks very much like the circle $x^2 + y^2 = 4$ that we found earlier by direct calculation.

This illustrates one of the main advantages of slope fields: *they allow us to visualize the qualitative behavior of solutions without explicitly solving the differential equation.*

■ Using a Slope Field

Consider the differential equation

$$y' = x^2 + y^2 - 1.$$

The slope at a point (x, y) is therefore

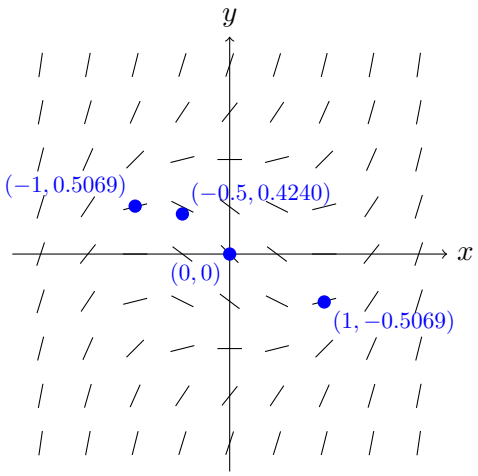
$$x^2 + y^2 - 1.$$

To sketch the solution satisfying the initial condition $y(0) = 0$, we first draw the slope field, and then we follow the directions indicated by the small line segments, starting from the point $(0, 0)$.

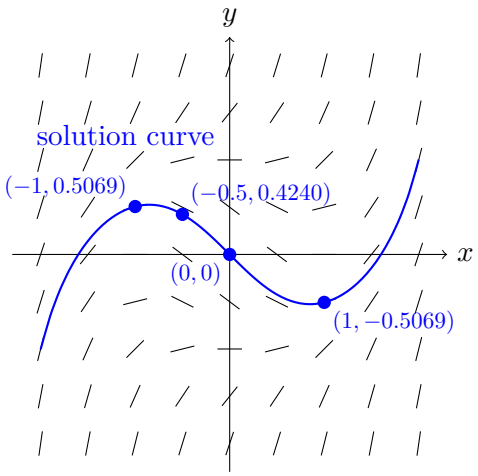
The following table lists some points on the corresponding solution curve, together with the slope at each of these points.

Point (x, y) on the solution curve	Slope $y' = x^2 + y^2 - 1$
$(-1, 0.5069)$	$(-1)^2 + 0.5069^2 - 1 \approx 0.257$
$(-0.5, 0.4240)$	$(-0.5)^2 + 0.4240^2 - 1 \approx -0.570$
$(0, 0)$	$0^2 + 0^2 - 1 = -1$
$(1, -0.5069)$	$1^2 + (-0.5069)^2 - 1 \approx 0.257$

The following figure shows the slope field, together with these four points.



Starting from the point $(0, 0)$ and following the directions indicated by the slope field gives the corresponding solution curve.



Remark. The more line segments we draw, the more accurately the slope field represents the possible solution curves. In practice, computers are particularly useful for producing detailed slope fields.

A Simple Electric Circuit

Slope fields can also be used to study differential equations arising from physical models.

Consider a simple electric circuit containing a resistor of resistance R, an inductor of inductance L, and a voltage source $E(t)$. If $I(t)$ denotes the current at time t, then *Kirchhoff's Law* gives

$$L\frac{dI}{dt} + RI = E(t).$$

For example, suppose

$$L = 4, \qquad R = 12, \qquad E(t) = 60.$$

Then

$$4\frac{dI}{dt} + 12I = 60,$$

so

$$\boxed{\frac{dI}{dt} = 15 - 3I}.$$

The sign of $\frac{dI}{dt}$ describes the behavior of the current:

1. If $I < 5$, then $15 - 3I > 0$, so the current increases.
2. If $I = 5$, then $15 - 3I = 0$, so the current remains constant.
3. If $I > 5$, then $15 - 3I < 0$, so the current decreases.

Thus, $I(t) = 5$ is an equilibrium solution, and the slope field indicates that the other solutions tend toward $I = 5$.

For example, the functions $I(t) = 5 - 5e^{-3t}$ and $I(t) = 5 + 3e^{-3t}$ are solutions corresponding to the initial conditions $I(0) = 0$ and $I(0) = 8$, respectively. Their graphs illustrate how the current approaches the equilibrium value $I = 5$.

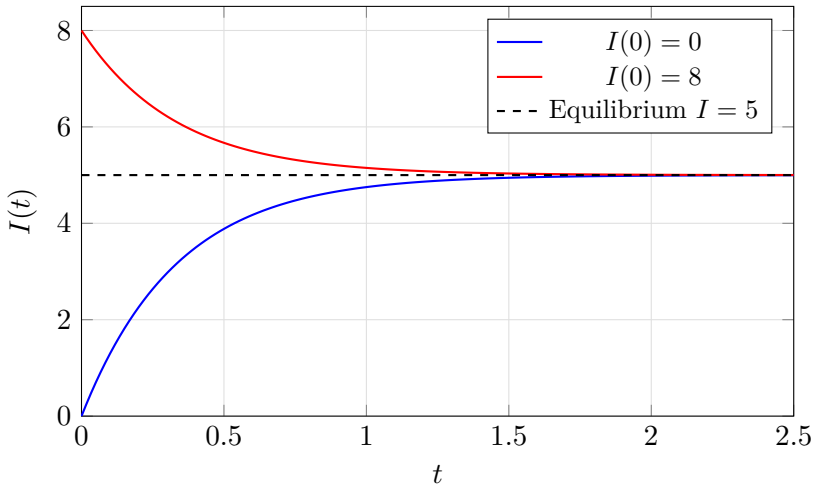


Figure 5: Solutions of the differential equation $I' = 15 - 3I$. Both solutions approach the equilibrium value $I = 5$ as $t \to +\infty$.

Autonomous Differential Equations

Notice that in the equation

$$\frac{dI}{dt} = 15 - 3I,$$

the independent variable t does not appear explicitly on the right-hand side.

Definition. Autonomous Differential Equation

A first-order differential equation of the form

$$y' = f(y)$$

is called an *autonomous differential equation.*

For an autonomous differential equation, the slope depends only on the value of y. Therefore, all the line segments along the same horizontal line in the slope field have the same slope.

This makes the qualitative behavior of autonomous differential equations particularly easy to visualize.

Euler's Method

Slope fields suggest a numerical method for approximating solutions of differential equations.

Consider the initial-value problem

$$\begin{cases} y' = F(x, y), \\ y(x_0) = y_0. \end{cases}$$

At the initial point (x_0, y_0), the slope of the solution curve is

$$F(x_0, y_0).$$

The idea behind Euler's Method comes from the tangent-line approximation. If (x_n, y_n) is close to the solution curve, then for a small step h,

$$y(x_n + h) \approx y(x_n) + hy'(x_n).$$

Since $y' = F(x, y)$, this suggests

$$y(x_n + h) \approx y(x_n) + hF(x_n, y(x_n)).$$

Replacing the exact value $y(x_n)$ by the approximation y_n leads to

$$y_{n+1} = y_n + hF(x_n, y_n).$$

In particular, starting from (x_0, y_0), the first approximation is

$$y_1 = y_0 + hF(x_0, y_0),$$

where

$$x_1 = x_0 + h.$$

We then repeat the same procedure starting from the new point (x_1, y_1):

$$y_2 = y_1 + hF(x_1, y_1),$$

with

$$x_2 = x_1 + h.$$

Continuing in this way gives Euler's Method.

Definition. Euler's Method

Consider the initial-value problem

$$\begin{cases} y' = F(x, y), \\ y(x_0) = y_0. \end{cases}$$

Choose a step size h and define

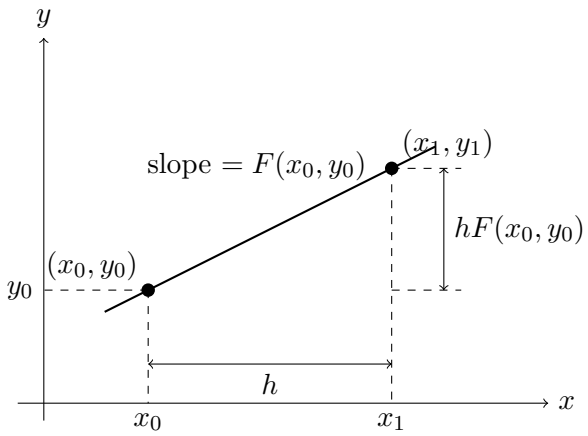
$$x_n = x_0 + nh.$$

Starting from y_0, Euler's Method defines successive approximations to the solution by

$$\boxed{y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})}, \qquad n \geq 1.$$

The number y_n is then used as an approximation of the exact value $y(x_n)$.

Geometrically, Euler's Method consists of following the tangent line for a short distance h, then recomputing the slope at the new point and changing direction accordingly.



■ Example of Euler's Method

Consider the initial-value problem

$$\begin{cases} y' = x + y, \\ y(0) = 1. \end{cases}$$

We apply Euler's Method with step size

$$h = 0.1.$$

Here,

$$F(x, y) = x + y,$$

and the recurrence relation becomes

$$y_n = y_{n-1} + 0.1(x_{n-1} + y_{n-1}).$$

Starting with

$$x_0 = 0, \qquad y_0 = 1,$$

we obtain

$$\begin{aligned} y_1 &= 1 + 0.1(0 + 1) = 1.1, \\ y_2 &= 1.1 + 0.1(0.1 + 1.1) = 1.22, \\ y_3 &= 1.22 + 0.1(0.2 + 1.22) = 1.362. \end{aligned}$$

Thus,

$$y(0.1) \approx 1.1, \qquad y(0.2) \approx 1.22, \qquad y(0.3) \approx 1.362.$$

Continuing the same procedure gives the following table:

n	x_n	y_n
0	0	1.000000
1	0.1	1.100000
2	0.2	1.220000
3	0.3	1.362000
4	0.4	1.528200
5	0.5	1.721020
6	0.6	1.943122
7	0.7	2.197434
8	0.8	2.487178
9	0.9	2.815895
10	1.0	3.187485

The exact solution of this initial-value problem is

$$y = 2e^x - x - 1.$$

The following graph compares the exact solution with the approximation obtained by Euler's Method.

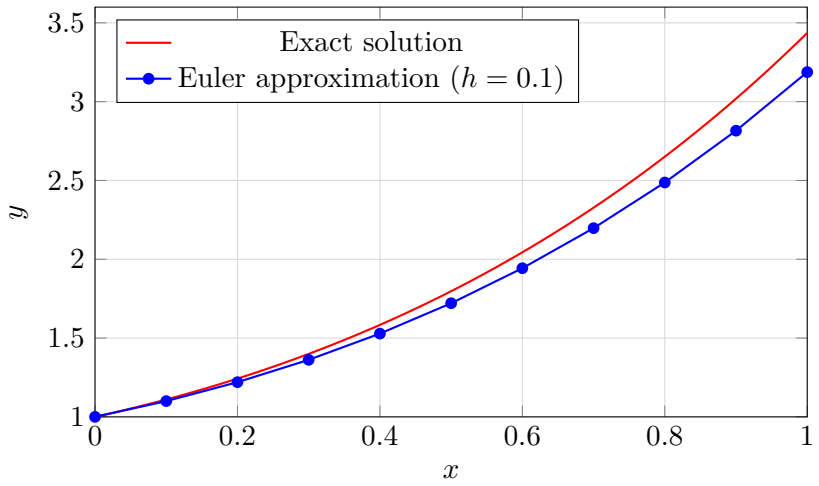


Figure 6: Comparison between the exact solution of $y' = x + y$, $y(0) = 1$, and Euler's Method with step size $h = 0.1$.

■ Accuracy of Euler's Method

Euler's Method gives an approximation rather than the exact solution. The accuracy depends strongly on the step size h.

A smaller step size means that the slope is recalculated more frequently, so the polygonal approximation generally follows the true solution curve more closely.

For example, for the initial-value problem

$$\begin{cases} y' = x + y, \\ y(0) = 1, \end{cases}$$

the Euler approximations become increasingly accurate as the step size decreases.

Warning. A smaller step size generally gives a better approximation, but it also requires more calculations.

■ Euler's Method for the Electric Circuit

Consider again the circuit equation

$$\frac{dI}{dt} = 15 - 3I$$

with the initial condition

$$I(0) = 0.$$

We use Euler's Method with step size

$$h = 0.1.$$

The recurrence relation is

$$I_n = I_{n-1} + 0.1(15 - 3I_{n-1}).$$

Starting from $I_0 = 0$, we obtain

$$\begin{aligned}
I_1 &= 0 + 0.1(15 - 3 \cdot 0) = 1.5, \\
I_2 &= 1.5 + 0.1(15 - 3 \cdot 1.5) = 2.55, \\
I_3 &= 2.55 + 0.1(15 - 3 \cdot 2.55) = 3.285, \\
I_4 &= 3.285 + 0.1(15 - 3 \cdot 3.285) = 3.7995, \\
I_5 &= 3.7995 + 0.1(15 - 3 \cdot 3.7995) = 4.15965.
\end{aligned}$$

Therefore, after half a second,

$$\boxed{I(0.5) \approx 4.16 \text{ A}}.$$

The exact solution of this initial-value problem is

$$I(t) = 5(1 - e^{-3t}).$$

The following graph shows the current $I(t)$ together with the values obtained by Euler's Method. The dashed line indicates the equilibrium current $I = 5$.

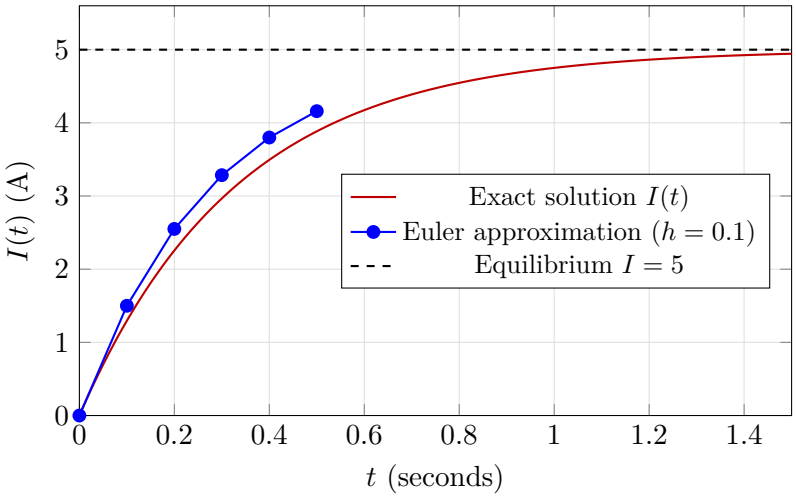


Figure 7: Current as a function of time for $I' = 15 - 3I$ with $I(0) = 0$. Euler's Method gives a piecewise linear approximation to the exact solution.