MAT 127 — Calculus C: Course Notes

These notes are based on the textbook *Calculus: Concepts and Contexts, 5th Edition, by James Stewart and Steve Kokoska*, on which the course is based. They provide a summary of the material covered during the class sessions. Students are strongly encouraged to supplement these notes by studying the textbook and practicing with the exercises provided there. It is also essential to attend office hours in order to ask questions and clarify any difficulties. Simply reading these notes is not sufficient, especially given the limited amount of time available during class. These notes will be updated after each week of classes.

Contents

1 Sequences

■ Definition

A sequence is a list of numbers: $a_1, a_2, a_3, a_4, \dots, a_n, \dots$

Some sequences begin with $n = 0$: $a_0, a_1, a_2, \dots, a_n, \dots$

The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example Fibonacci sequence

The Fibonacci sequence $\{f_n\}$ is defined recursively by

$$f_1 = 1, \qquad f_2 = 1, \qquad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$.

■ Limit of a Sequence

Let us consider the example $a_n = \frac{n}{n+1}$. By dividing the numerator and the denominator by n, we get:

$$\frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

and $\frac{1}{n}$ becomes smaller and smaller as n tends to 0. In this case, we write $\lim_{n\to+\infty}\frac{1}{n}=0$. Thus, we have:

$$\lim_{n\to+\infty}\frac{n}{n+1}=\frac{1}{1+0}=1.$$

In general, the notation $\lim_{n\to\infty} a_n = L$ means that the terms of the sequence a_n approach L as n becomes large.

Definition. Limit of a Sequence

A sequence $\{a_n\}$ has the limit L, written as

$$\lim_{n\to+\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as } n\to+\infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, then we say the sequence $\{a_n\}$ *converges*, or is *convergent*. Otherwise, we say the sequence *diverges*, or is *divergent*.

Limit of a Function and Corresponding Sequence

Recall that, for a function f, the statement

$$\lim_{x\to+\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

The definitions of the limit of a sequence and the limit of a function are therefore very similar. The main difference between

$$\lim_{n\to+\infty} a_n = L \qquad \text{and} \qquad \lim_{x\to+\infty} f(x) = L$$

is that n is restricted to integer values.

Theorem. Limit of a Function and Corresponding Sequence

If $\lim_{x\to+\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to+\infty} a_n = L$.

For example, we know that:

$$\lim_{x\to+\infty}\frac{1}{x^r}=0 \qquad \text{when } r>0.$$

By applying the theorem with the function $f(x)=\frac{1}{x^r}$ and $a_n = f(n)$, we have that $\lim_{n\to+\infty} a_n = \lim_{x\to+\infty} f(x) = 0$. Thus, we obtain:

$$\boxed{\lim_{n\to+\infty}\frac{1}{n^r}=\lim_{n\to+\infty} a_n = 0 \qquad \text{if } r>0.}$$

Infinite Limits

If a_n becomes arbitrarily large as n increases without bound, we write

$$\lim_{n\to+\infty} a_n = +\infty.$$

In this case, we do not consider the sequence a_n to be convergent. We say that the sequence diverges, or is divergent, just like sequences that do not have a limit.

For example, the sequence $a_n = n$ satisfies:

$$\lim_{n\to+\infty} n = +\infty.$$

Warning: Not Every Sequence Has a Limit

Not every sequence has a limit. For example, consider the sequence

$$a_n = (-1)^n.$$

Its terms oscillate:

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}.$$

Since the terms do not approach a single number L as $n \to +\infty$, the sequence is divergent.

Limit Laws

The Limit Laws for functions also hold for the limits of sequences.

Theorem. Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

1. $$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

2. $$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

3. $$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n.$$

4. $$\lim_{n\to\infty} c = c.$$

5. $$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

6. $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \qquad \text{if } \lim_{n\to\infty} b_n \neq 0.$$

7. $$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p, \qquad \text{if } p > 0 \text{ and } a_n > 0.$$

Squeeze Theorem

Theorem. Squeeze Theorem for Sequences

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Corollary.

If $\lim_{n\to\infty} |c_n| = 0$, then $\lim_{n\to\infty} c_n = 0$.

Proof. For every n, we have $-|a_n| \le c_n \le |a_n|$. Since $\lim_{n\to+\infty}(-|c_n|) = 0$ and $\lim_{n\to+\infty} |c_n| = 0$, the Squeeze Theorem, applied with $a_n = -|c_n|$ and $b_n = |c_n|$, gives $\lim_{n\to+\infty} c_n = 0$. □

Example Use the Absolute Value

Let us compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$. Consider the sequence of absolute values:

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, by the (Corollary of the) Squeeze Theorem,

$$\boxed{\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.}$$

■ Recall: L'Hôpital's Rule

Recall that L'Hôpital's Rule can be used to evaluate certain indeterminate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable functions, where $I = [A, +\infty[$. Suppose that $g'(x) \neq 0$ for every $x > A$ and that

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

where L is finite or infinite. If either

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = 0$$

or

$$\lim_{x\to+\infty} |f(x)| = \lim_{x\to+\infty} |g(x)| = +\infty,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L.$$

Example Applying L'Hôpital's Rule to a Sequence

Let us compute $\lim_{n\to+\infty} \frac{\ln n}{n}$. Since L'Hôpital's Rule applies to functions of a real variable rather than directly to sequences, consider the corresponding function

$$\frac{f(x)}{g(x)} = \frac{\ln x}{x},$$

where $f(x) = \ln x$ and $g(x) = x$.

We have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln x = +\infty$ and $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} x = +\infty$. Therefore, the quotient $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{\infty}{\infty}$. By L'Hôpital's Rule,

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{1/x}{1} = 0.$$

Therefore,

$$\boxed{\lim_{n\to+\infty} \frac{\ln n}{n} = 0.}$$

■ Sequence–Function Links

Recall that a continuous function is, informally, a function whose graph can be drawn "without lifting your pencil." In other words, the graph has no jumps or breaks.

This amounts to saying that f is continuous at every point of its domain. Recall that f is continuous at a point x_0 if the value of the function at x_0 agrees with the limiting value of the function as x approaches x_0:

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Therefore, if a sequence $\{a_n\}$ converges to L and f is continuous at L, then the values $f(a_n)$ converge to $f(L)$.

Theorem. Limit of a Continuous Function Applied to a Sequence

If $\lim_{n\to+\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to+\infty} f(a_n) = f(L)$.

Example Use a Continuous Function

Let us compute $\displaystyle\lim_{n\to+\infty} \sin\left(\frac{\pi}{n}\right)$. We have $\lim_{n\to+\infty} \frac{\pi}{n} = 0$, and the sine function is continuous at 0. Since $\sin(0) = 0$, the theorem above gives

$$\lim_{n\to+\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to+\infty} \frac{\pi}{n}\right) = \sin 0 = 0.$$

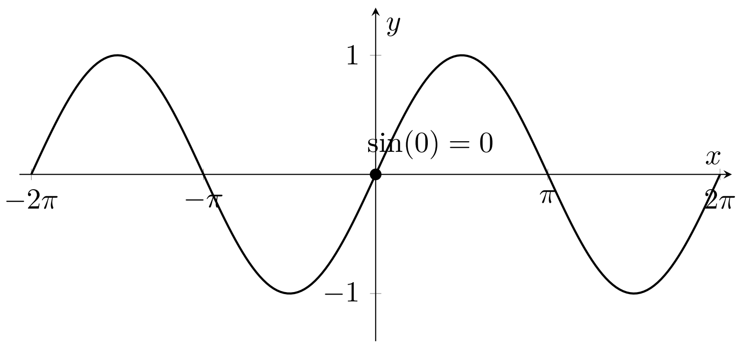


Figure 1: Graph of the function $f(x) = \sin x$.

Geometric Sequences and Their Limits

Definition. Geometric sequences

Let $r \in \mathbb{R}$ be a real number. A sequence of the form

$$a_n = r^n$$

is called a geometric sequence with common ratio r.

Proposition. Limit of the Sequence $\{r^n\}$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent otherwise. More precisely,

$$\lim_{n\to+\infty} r^n = \begin{cases} 0, & -1 < r < 1, \\ 1, & r = 1. \end{cases}$$

Proof. We consider several cases.

- If $r > 1$, then
$$\lim_{n\to+\infty} r^n = +\infty,$$
so the sequence diverges.
- If $0 < r < 1$, then
$$\lim_{n\to+\infty} r^n = 0,$$
so the sequence converges to 0.

- For $r = 0$ and $r = 1$, we have

$$\lim_{n\to+\infty} 0^n = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} 1^n = 1.$$

- Now suppose that $-1 < r < 0$. Then $0 < |r| < 1$, and therefore

$$\lim_{n\to+\infty} |r^n| = \lim_{n\to+\infty} |r|^n = 0.$$

  Since

$$-|r|^n \le r^n \le |r|^n,$$

  the Squeeze Theorem gives

$$\lim_{n\to+\infty} r^n = 0.$$

- If $r = -1$, the sequence is

$$\{-1, 1, -1, 1, -1, 1, \ldots\},$$

  so it oscillates between -1 and 1 and is divergent.

- Finally, if $r < -1$, the terms alternate between positive and negative values while their absolute values increase without bound. Hence, the sequence is divergent. □

■ Monotonic Sequences

Recall that for a function f defined on an interval I:

Strict monotonicity

- f is *strictly increasing* on I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- f is *strictly decreasing* on I if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Nonstrict monotonicity

- f is *nondecreasing* on I if $x_1 < x_2$ implies $f(x_1) \le f(x_2)$.
- f is *nonincreasing* on I if $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$.

A function is called *monotonic* if it is either nondecreasing or nonincreasing.

Recall that if f is differentiable, then $f'(x) > 0$ implies that f is strictly increasing, while $f'(x) < 0$ implies that f is strictly decreasing. Similarly, $f'(x) \ge 0$ implies that f is nondecreasing, while $f'(x) \le 0$ implies that f is nonincreasing.

The same terminology applies to sequences by comparing two consecutive terms a_n and a_{n+1}.

Definition. Increasing, Decreasing, and Monotonic Sequences

Strict monotonicity

- A sequence $\{a_n\}$ is *(strictly) increasing* if $a_n < a_{n+1}$ for all $n \ge 1$.
- A sequence $\{a_n\}$ is *(strictly) decreasing* if $a_n > a_{n+1}$ for all $n \ge 1$.

Nonstrict monotonicity

- A sequence $\{a_n\}$ is *nondecreasing* if $a_n \le a_{n+1}$ for all $n \ge 1$.
- A sequence $\{a_n\}$ is *nonincreasing* if $a_n \ge a_{n+1}$ for all $n \ge 1$.

A sequence is *monotonic* if it is either nondecreasing or nonincreasing.

Example

1. Consider the sequence
$$a_n = n.$$
We have
$$a_{n+1} = n + 1 > n = a_n.$$
Therefore, the sequence $\{n\}$ is increasing.

2. Consider the sequence
$$a_n = \frac{1}{n}.$$
We have
$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n.$$
Therefore, the sequence $\left\{\frac{1}{n}\right\}$ is decreasing.

3. Consider the sequence
$$a_n = \ln(n + 1).$$
Let $f : [1, +\infty[\to \mathbb{R}$ be defined by
$$f(x) = \ln(x + 1).$$
We have
$$f'(x) = \frac{1}{x+1} > 0$$
for every $x \geq 1$. Therefore, f is increasing on $[1, +\infty[$. Since $a_n = f(n)$, the sequence $\{a_n\}$ is increasing.

Bounded Sequences

Definition. Bounded Sequence

A sequence $\{a_n\}$ is *bounded above* if there is a number M such that
$$a_n \leq M \qquad \text{for all } n \geq 1.$$
It is *bounded below* if there is a number m such that
$$m \leq a_n \qquad \text{for all } n \geq 1.$$
If it is bounded both above and below, then $\{a_n\}$ is a *bounded sequence*. Equivalently, $\{a_n\}$ is bounded if there exists a number $C > 0$ such that
$$|a_n| \leq C \qquad \text{for all } n \geq 1.$$

Warning. A bounded sequence is not necessarily convergent: for instance, $(-1)^n$ is bounded but divergent. Similarly, a monotonic sequence is not necessarily convergent: for instance, $a_n = n$ is increasing but divergent. However, if a sequence is both bounded and monotonic, then it is convergent.

Theorem. Monotonic Sequence Theorem

Every bounded and monotonic sequence is convergent.

2 Series

Definition

We would like now to add the terms of an infinite sequence $\{a_n\}_{n=1}^{+\infty}$ as an expression of the form:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Such an infinite sum is called an infinite *series* and is denoted by

$$\sum_{i=1}^{+\infty} a_i \qquad \text{or} \qquad \sum a_n.$$

For a general series, define the partial sums

$$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad s_3 = a_1 + a_2 + a_3, \qquad \ldots$$

and in general

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to+\infty} s_n = s$ exists as a finite number, then we call s the sum of the infinite series $\sum a_n$.

Definition. Convergent Series

Given a series

$$\sum_{n=1}^{+\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

let s_n denote the nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i.$$

If the sequence $\{s_n\}$ converges and

$$\lim_{n\to+\infty} s_n = s$$

exists as a real number, then the series $\sum a_n$ is called *convergent* and we write

$$\sum_{n=1}^{+\infty} a_n = s.$$

The number s is called the sum of the series. If $\{s_n\}$ diverges, then the series is called divergent. In the convergent case, we have

$$\sum_{n=1}^{+\infty} a_n = \lim_{n\to+\infty} \sum_{i=1}^{n} a_i.$$

Warning. Not every infinite series is convergent. For example,

$$\sum_{n=1}^{+\infty} 1 = 1 + 1 + 1 + \cdots$$

is divergent because its sequence of partial sums is $s_n = n$, and

$$\lim_{n\to+\infty} s_n = +\infty.$$

Another important example is the *harmonic series* (see below in the document)

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

which is also divergent with $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

Geometric Series

Theorem. The Geometric Series

Let $a \neq 0$. The geometric series

$$\sum_{n=1}^{+\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$, and

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$, then the geometric series is divergent.
In short,

$$\boxed{\text{sum of a convergent geometric series} = \frac{\text{first term}}{1 - \text{common ratio}}.}$$

Proof. Let

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

be the nth partial sum.

- If $r = 1$, then

  $$s_n = na.$$

  Since $a \neq 0$, the limit $\lim_{n\to\infty} s_n$ is not finite. Therefore, the series diverges.

- If $r \neq 1$, then

  $$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

  Subtracting,

  $$s_n - rs_n = a - ar^n,$$

  and therefore

  $$s_n = \frac{a(1-r^n)}{1-r}.$$

- If $|r| < 1$, then

  $$\lim_{n\to+\infty} r^n = 0.$$

  Hence,

  $$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

  Therefore,

  $$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}.$$

- If $r = -1$, then $r^n = (-1)^n$ is divergent, so $\{s_n\}$ is divergent.

- If $|r| > 1$, then $\{r^n\}$ is divergent, so $\{s_n\}$ is divergent.

Therefore, the geometric series converges exactly when $|r| < 1$. $\square$

Example

Consider the series
$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$
This is a geometric series with first term $a = 5$ and common ratio $r = -\frac{2}{3}$. Since $|r| < 1$, the series converges. Using the formula for the sum of a convergent geometric series, we obtain
$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-(-2/3)} = 3.$$
Therefore, the sum of the series is 3. The partial sums oscillate around 3 and approach 3 as the number of terms increases.

■ Sum of the First n Positive Integers

Proposition. Sum of the First n Positive Integers

For every integer $n \geq 1$,
$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. Let
$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$
Writing the same sum in reverse order gives
$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$
Adding the two expressions term by term, we obtain
$$2S = (n+1) + (n+1) + \cdots + (n+1).$$
There are n terms on the right-hand side, so
$$2S = n(n+1).$$
Therefore,
$$S = \frac{n(n+1)}{2}.$$
Hence,
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$
□

■ A Geometric Series with Variable Terms

Consider the series
$$\sum_{n=0}^{+\infty} x^n, \qquad |x| < 1.$$
This is a geometric series with first term $a = 1$ and common ratio $r = x$. Since $|x| < 1$, the geometric series converges and
$$\boxed{\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1.}$$
□

Telescoping Series

Consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}.$$

Using the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1},$$

the nth partial sum is

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right).$$

Expanding the sum gives

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

The intermediate terms cancel, so

$$s_n = 1 - \frac{1}{n+1}.$$

A sum in which consecutive terms cancel in this way is called a *telescoping sum*. Taking the limit of the partial sums,

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore, the series is convergent and

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1.}$$

Notice that the terms satisfy $\lim_{n\to+\infty} a_n = 0$, while the partial sums satisfy $\lim_{n\to+\infty} s_n = 1$.

The Harmonic Series

The harmonic series is

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

We show that it is divergent by considering the partial sums $s_2, s_4, s_8, s_{16}, \ldots$.

Proof. First,

$$s_2 = 1 + \frac{1}{2}.$$

Next,

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}.$$

Also,

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}.$$

Similarly,

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) > 1 + \frac{4}{2}.$$

Continuing in the same way,

$$s_{32} > 1 + \frac{5}{2}, \qquad s_{64} > 1 + \frac{6}{2},$$

and in general,

$$s_{2^n} > 1 + \frac{n}{2}.$$

Since

$$\lim_{n\to+\infty} \left(1 + \frac{n}{2}\right) = +\infty,$$

the partial sums s_{2^n} become arbitrarily large. Therefore, the sequence of partial sums $\{s_n\}$ is divergent, and hence

$$\sum_{n=1}^{+\infty} \frac{1}{n}$$

is divergent. □

■ Convergent Series Behavior

The next theorem describes the behavior of the terms of a convergent series.

Theorem. Limit of the Terms of a Convergent Series

If the series $\sum_{n=1}^{+\infty} a_n$ is convergent, then

$$\lim_{n\to+\infty} a_n = 0.$$

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$ be the nth partial sum. Then

$$a_n = s_n - s_{n-1}.$$

Since $\sum a_n$ is convergent, the sequence of partial sums $\{s_n\}$ is convergent. Let

$$\lim_{n\to+\infty} s_n = s.$$

We also have

$$\lim_{n\to+\infty} s_{n-1} = s.$$

Therefore,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} (s_n - s_{n-1}) = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} s_{n-1} = s - s = 0.$$

□

Warning. The converse of the theorem is false. The condition

$$\lim_{n\to+\infty} a_n = 0$$

does not imply that $\sum a_n$ is convergent. For example, for the harmonic series

$$\sum_{n=1}^{+\infty} \frac{1}{n},$$

we have

$$\lim_{n\to+\infty} \frac{1}{n} = 0,$$

but the series is divergent.

The following divergence test is the contrapositive of the preceding theorem.

Theorem. The Test for Divergence

If $\lim_{n\to+\infty} a_n$ does not exist, or if

$$\lim_{n\to+\infty} a_n \neq 0,$$

then the series

$$\sum_{n=1}^{+\infty} a_n$$

is divergent.

Warning. If $\lim_{n\to+\infty} a_n = 0$, then we cannot conclude whether the series $\sum a_n$ converges or diverges. The series may converge, or it may diverge. Further analysis is required.

Combining Series

Theorem. Properties of Convergent Series

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$, where c is a constant, $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$. Moreover,

$$\text{(i)} \qquad \sum_{n=1}^{+\infty} ca_n = c\sum_{n=1}^{+\infty} a_n,$$

$$\text{(ii)} \qquad \sum_{n=1}^{+\infty}(a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n,$$

$$\text{(iii)} \qquad \sum_{n=1}^{+\infty}(a_n - b_n) = \sum_{n=1}^{+\infty} a_n - \sum_{n=1}^{+\infty} b_n.$$

These properties of convergent series follow from the corresponding Limit Laws for Sequences.

Example

Let us compute

$$\sum_{n=1}^{+\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right).$$

The series $\sum_{n=1}^{+\infty} \frac{1}{2^n}$ is a geometric series with first term $a = \frac{1}{2}$ and common ratio $r = \frac{1}{2}$. Therefore,

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

We also know that

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1.$$

Using the properties of convergent series, we obtain

$$\sum_{n=1}^{+\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{+\infty} \frac{1}{2^n} = 3 \cdot 1 + 1 = \boxed{4}.$$

Property. Adding or Removing Finitely Many Terms

Adding or removing a finite number of terms does not affect the convergence or divergence of a series. More generally, for any integer $N \geq 1$, the series

$$\sum_{n=1}^{+\infty} a_n$$

and

$$\sum_{n=N+1}^{+\infty} a_n$$

have the same convergence behavior.

For example, the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

and

$$\sum_{n=100}^{+\infty} \frac{1}{n^2}$$

have the same convergence behavior. Indeed,

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n=1}^{99} \frac{1}{n^2} + \sum_{n=100}^{+\infty} \frac{1}{n^2},$$

and the first sum on the right-hand side contains only finitely many terms. Therefore, one series converges if and only if the other converges.

For your information, a famous result due to Euler states that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Thus, in particular, both series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ and $\sum_{n=100}^{+\infty} \frac{1}{n^2}$ are convergent. Dans le prochain chapitre on verra comment montrer la convergence de $\sum_{n=1}^{+\infty} \frac{1}{n^2}$

3 The Integral and Comparison Tests; Estimating Sums

■ The Integral Test

Let us consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

Set

$$f(x) = \frac{1}{x^2}.$$

The function f is continuous, positive, and decreasing on $[1, +\infty[$. For each integer $n \geq 2$, we draw a rectangle of width 1 on the interval $[n-1, n]$ and height $f(n) = \frac{1}{n^2}$. Since f is decreasing, each rectangle lies below the curve $y = \frac{1}{x^2}$ (see Figure 2 below). Therefore, for every $N \geq 2$,

$$\begin{aligned}
\sum_{n=2}^{N} \frac{1}{n^2} &= \text{sum of the areas of the rectangles } [1,2] \times \left[0, \frac{1}{2^2}\right],\ [2,3] \times \left[0, \frac{1}{3^2}\right], \dots,\ [N-1, N] \times \left[0, \frac{1}{N^2}\right] \\
&\leq \int_1^N \frac{1}{x^2}\,dx \qquad \text{because each rectangle lies below the graph of } f(x) = \frac{1}{x^2} \\
&= \left[\frac{-1}{x}\right]_{x=1}^{x=N} \qquad \text{because an antiderivative of } \frac{1}{x^2} \text{ is } -\frac{1}{x} \\
&= -\frac{1}{N} - (-1) \qquad \text{by evaluating the antiderivative at } N \text{ and } 1 \\
&= 1 - \frac{1}{N} < 1 \qquad \text{because } N \geq 2.
\end{aligned}$$

It follows that

$$\sum_{n=1}^{N} \frac{1}{n^2} = 1 + \sum_{n=2}^{N} \frac{1}{n^2} < 2.$$

Since all the terms are positive, the sequence of partial sums $s_N = \sum_{n=1}^{N} \frac{1}{n^2}$ is increasing. Since it is also bounded above by 2, the sequence s_N is convergent. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

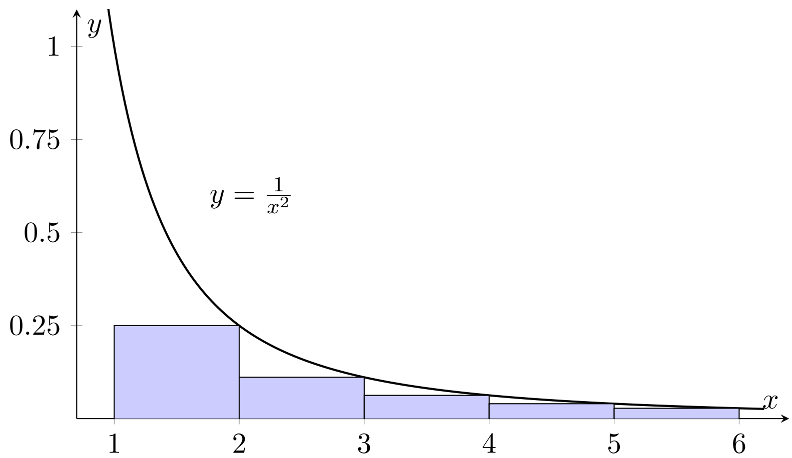


Figure 2: The rectangles of heights $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}, \frac{1}{6^2}$ lie below the graph of $y = \frac{1}{x^2}$. Their total area is therefore smaller than $\int_1^6 \frac{1}{x^2}\,dx$.

This geometric comparison between sums and integrals leads to the following theorem.

Theorem. The Integral Test

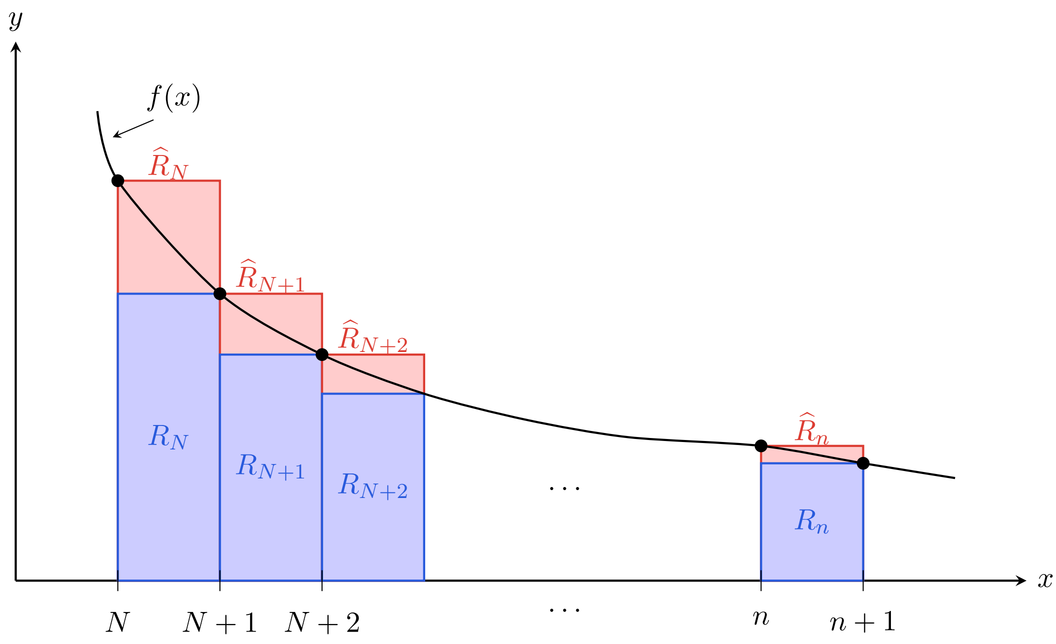
Let $N \geq 1$ and suppose that $f : [N, +\infty[\to \mathbb{R}$ is continuous, positive, and decreasing. Let $a_n = f(n)$. Then, for every $n \geq N$,

$$\int_N^{n+1} f(x)\ dx \leq \sum_{k=N}^{n} f(k) \leq f(N) + \int_N^n f(x)\ dx.$$

In particular, the series $\sum_{n=N}^{+\infty} a_n$ and the sequence $\{\int_N^n f(x)\ dx\}_{n=N}^{+\infty}$ have the same convergence behavior: either both converge or both diverge. In the convergent case, if $L = \lim\limits_{n\to+\infty} \int_N^n f(x)\ dx$, then we have:

$$L \leq \sum_{n=N}^{+\infty} f(n) \leq f(N) + L.$$

Proof. Let R_k be the rectangle with base $[k, k+1]$ and height $f(k+1)$, and let $\widehat{R}_k$ be the rectangle with base $[k, k+1]$ and height $f(k)$.



Since f is decreasing and positive, the area under the graph of f over $[k, k+1]$ lies between the blue area $\text{Area}(R_k)$ and the red area $\text{Area}(\widehat{R}_k)$ of these two rectangles. Therefore,

$$\text{Area}(R_k) \leq \int_k^{k+1} f(x)\, dx \leq \text{Area}(\widehat{R}_k).$$

Since both rectangles have width 1, the area of the smaller rectangle is $\text{Area}(R_k) = 1 \cdot f(k+1) = f(k+1)$, while the area of the larger rectangle is $\text{Area}(\widehat{R}_k) = 1 \cdot f(k) = f(k)$. We thus deduce that:

$$f(k+1) \leq \int_k^{k+1} f(x)\, dx \leq f(k).$$

Summing these inequalities for $k = N, \ldots, n$, we obtain

$$\sum_{k=N}^{n} f(k+1) \leq \int_N^{n+1} f(x)\, dx \leq \sum_{k=N}^{n} f(k).$$

Equivalently,

$$\int_N^{n+1} f(x)\, dx \leq \sum_{k=N}^{n} f(k) \leq f(N) + \int_N^n f(x)\, dx.$$

Set $s_n := \sum_{k=N}^{n} f(k)$. Since $f(k) > 0$, the sequence (s_n) is increasing.

• Suppose first that $\{\int_N^n f(x)\,dx\}_{n=N}^{+\infty}$ converges. Then the sequence $\{\int_N^n f(x)\,dx\}_{n=N}^{+\infty}$ is bounded. By the inequality above, $s_n \le f(N) + \int_N^n f(x)\,dx$, so $\{s_n\}$ is bounded above. Since $\{s_n\}$ is increasing and bounded above, it converges. Hence $\sum_{n=N}^{+\infty} f(n)$ converges.

• Conversely, suppose that $\sum_{n=N}^{+\infty} f(n)$ converges. Then the sequence $\{s_n\}$ is bounded. Since $\int_N^{n+1} f(x)\,dx \le s_n$, the sequence $\{\int_N^{n+1} f(x)\,dx\}_{n=N}^{+\infty}$ is bounded above. Moreover, since $f(x) > 0$, this sequence is increasing. Therefore it converges and hence $\{\int_N^n f(x)\,dx\}_{n=N}^{+\infty}$ converges. □

Example Using the Integral Test

Let us determine whether the series

$$\sum_{n=1}^{+\infty} \frac{\ln n}{n}$$

converges or diverges. Consider

$$f(x) = \frac{\ln x}{x}.$$

The function is positive and continuous for $x > 1$. Moreover,

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Since $f'(x) < 0$ for $x > e$, the function is decreasing for all sufficiently large x. We now consider the improper integral:

$$\int_1^{+\infty} \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \left[\frac{(\ln x)^2}{2}\right]_{x=1}^{x=t} = \lim_{t\to+\infty} \left(\frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2}\right) = +\infty.$$

Therefore, by the Integral Test, $\sum_{n=1}^{+\infty} \frac{\ln n}{n}$ is divergent.

■ p-Series

Theorem. The p-Series

The series

$$\sum_{n=1}^{+\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $p \le 1$.

Proof. If $p \le 0$, then $\lim_{n\to+\infty} \frac{1}{n^p} \ne 0$, so the series diverges by the Test for Divergence. Suppose now that $p > 0$. Consider

$$f(x) = \frac{1}{x^p}.$$

This function is continuous, positive, and decreasing on $[1, +\infty[$. If $p = 1$, then

$$\int_1^{+\infty} \frac{1}{x}\,dx = +\infty,$$

so the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges. If $p \ne 1$, then

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{t^{1-p} - 1}{1 - p}.$$

This limit is finite exactly when $p > 1$. Therefore, by the Integral Test, the p-series converges exactly when $p > 1$. □

Example Examples of p-Series

1. The series $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ is convergent because $p = 3 > 1$.
2. The series $\sum_{n=1}^{+\infty} \frac{1}{n^{1/3}}$ is divergent because $p = \frac{1}{3} \le 1$.

■ The Comparison Test

The idea of the Comparison Test is to compare a series with a simpler series whose convergence behavior is already known.

Theorem. The Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms: $a_n \ge 0$ and $b_n \ge 0$.

- If $a_n \le b_n$ for all sufficiently large n and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $a_n \ge b_n$ for all sufficiently large n and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

Two particularly useful series for comparison are the p-series

$$\sum_{n=1}^{+\infty} \frac{1}{n^p}$$

and the geometric series

$$\sum_{n=1}^{+\infty} ar^{n-1}.$$

Example Comparison with a p-Series

Consider the series

$$\sum_{n=1}^{+\infty} \frac{5}{2n^2 + 4n + 3}.$$

We have

$$0 < \frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2} = \frac{5}{2} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2$, the Comparison Test implies that

$$\sum_{n=1}^{+\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent.

Example Comparison with the Harmonic Series

Consider the series

$$\sum_{n=1}^{+\infty} \frac{\ln n}{n}.$$

For $n \ge 3$, we have $\ln n > 1$, and therefore

$$\frac{\ln n}{n} > \frac{1}{n}.$$

Since the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent, the Comparison Test implies that

$$\sum_{n=1}^{+\infty} \frac{\ln n}{n}$$

is divergent.

Warning. The direction of the inequality is essential. To prove convergence, we compare with a larger convergent series. To prove divergence, we compare with a smaller divergent series. For example,

$$\frac{1}{2^n - 1} > \frac{1}{2^n},$$

but the fact that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges does not allow us to conclude anything from the Comparison Test.

■ The Limit Comparison Test

When two series have terms with approximately the same behavior for large n, the Limit Comparison Test is often more convenient than finding an explicit inequality.

Theorem. The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with terms $a_n > 0$ and $b_n > 0$. If

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = c,$$

where $0 < c < +\infty$, then the two series $\sum a_n$ and $\sum b_n$ have the same convergence behavior: either both converge or both diverge.

Example Using the Limit Comparison Test

Consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{2^n - 1}.$$

Let

$$a_n = \frac{1}{2^n - 1} \qquad \text{and} \qquad b_n = \frac{1}{2^n}.$$

Then

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{2^n}{2^n - 1} = \lim_{n\to+\infty} \frac{1}{1 - \frac{1}{2^n}} = 1.$$

Since $\sum_{n=1}^{+\infty} \frac{1}{2^n}$ is a convergent geometric series and the limit is positive and finite, the Limit Comparison Test implies that $\sum_{n=1}^{+\infty} \frac{1}{2^n-1}$ is convergent.

Estimating the Sum of a Series

Definition. Remainder of a Convergent Series

Suppose that a convergent series

$$\sum_{n=1}^{+\infty} a_n$$

has sum s, and let

$$s_n = \sum_{k=1}^{n} a_k$$

be its nth partial sum. The *remainder* after n terms is defined by

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots .$$

Thus, R_n is the error made when s_n is used to approximate s.

Theorem. Remainder Estimate for the Integral Test

Suppose f is continuous, positive, and decreasing, $a_n = f(n)$, and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{+\infty} f(x)\,dx \le R_n \le \int_{n}^{+\infty} f(x)\,dx.$$

In particular,

$$R_n \le \int_{n}^{+\infty} f(x)\,dx$$

gives an upper bound for the error when s_n is used to approximate the sum.

Example Estimating the Sum of a Series

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Let

$$f(x) = \frac{1}{x^3}.$$

We have

$$\int_{n}^{+\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}.$$

Using the first 10 terms gives

$$s_{10} = \sum_{n=1}^{10} \frac{1}{n^3} \approx 1.1975.$$

The error satisfies

$$R_{10} \le \int_{10}^{+\infty} \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = 0.005.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.1975$$

with an error of at most 0.005.

Suppose now that we want the error to be at most 0.0005. We need

$$\frac{1}{2n^2} \le 0.0005.$$

Thus,

$$n^2 \geq 1000$$

and

$$n \geq \sqrt{1000} \approx 31.623.$$

Therefore, at least 32 terms are required.

Property. Lower and Upper Bounds for the Sum

Under the assumptions of the Integral Test,

$$s_n + \int_{n+1}^{+\infty} f(x)\,dx \leq s \leq s_n + \int_n^{+\infty} f(x)\,dx.$$

Example An Improved Estimate

For

$$\sum_{n=1}^{+\infty} \frac{1}{n^3},$$

using $n = 10$ gives

$$s_{10} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}.$$

Numerically,

$$1.201664 \leq s \leq 1.202532.$$

Using the midpoint of this interval,

$$s \approx \frac{1.201664 + 1.202532}{2} \approx 1.2021,$$

with an error smaller than 0.0005.

Property. Comparing Remainders

Suppose $0 \leq a_n \leq b_n$ and both series converge. If

$$R_n = \sum_{k=n+1}^{+\infty} a_k \qquad \text{and} \qquad T_n = \sum_{k=n+1}^{+\infty} b_k,$$

then

$$0 \leq R_n \leq T_n.$$

Thus, an error estimate for $\sum b_n$ also gives an error estimate for $\sum a_n$.

Example Estimating an Error by Comparison

Consider

$$\sum_{n=1}^{+\infty} \frac{1}{n^3 + 1}.$$

Since

$$0 < \frac{1}{n^3 + 1} < \frac{1}{n^3},$$

the series converges by the Comparison Test. Moreover, the remainder after n terms satisfies

$$R_n \leq \int_n^{+\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}.$$

For $n = 100$,

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005.$$

Using the first 100 terms,

$$\sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538.$$

Therefore,

$$\sum_{n=1}^{+\infty} \frac{1}{n^3+1} \approx 0.6864538$$

with an error smaller than 0.00005.

4 Other Convergence Tests

Alternating Series

Definition. Alternating Series

An *alternating series* is a series whose terms alternate between positive and negative values. It can be written in one of the forms

$$\sum_{n=1}^{+\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

or

$$\sum_{n=1}^{+\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots,$$

where $b_n \geq 0$.

For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n}$$

is an alternating series.

Theorem. Alternating Series Test

Consider an alternating series

$$\sum_{n=1}^{+\infty}(-1)^{n-1}b_n,$$

where $b_n \geq 0$. If

1. $b_{n+1} \leq b_n$ for all sufficiently large n;
2. $\lim_{n\to+\infty} b_n = 0$;

then the series is convergent.

Warning. Both conditions are important. Moreover, the Alternating Series Test can only prove convergence. If one of the conditions fails, the test is inconclusive and we must use another argument to determine whether the series converges or diverges.

The Alternating Harmonic Series

Consider the alternating harmonic series

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Let $b_n = \frac{1}{n} \geq 0$. We check the two conditions of the Alternating Series Test:

1. Since $\frac{1}{n+1} < \frac{1}{n}$, we have $b_{n+1} < b_n$.
2. We have $\lim_{n\to+\infty} b_n = \lim_{n\to+\infty}\frac{1}{n} = 0$.

Therefore, $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}$ is convergent by the Alternating Series Test.

■ When the Alternating Series Test Does Not Apply

Consider the series

$$\sum_{n=1}^{+\infty}(-1)^n\frac{3n}{4n-1}.$$

Let

$$b_n=\frac{3n}{4n-1}\geq 0.$$

We have

$$\lim_{n\to+\infty}b_n=\lim_{n\to+\infty}\frac{3n}{4n-1}=\lim_{n\to+\infty}\frac{3}{4-\frac{1}{n}}=\frac{3}{4}\neq 0.$$

Therefore, the second condition of the Alternating Series Test is not satisfied. Actually,

$$\lim_{n\to+\infty}(-1)^n\frac{3n}{4n-1}$$

does not exist, so the terms of the series do not tend to 0. Hence, by the Test for Divergence,

$$\sum_{n=1}^{+\infty}(-1)^n\frac{3n}{4n-1}$$

is divergent.

■ Another Application of the Alternating Series Test

Consider the series

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n^2}{n^3+1}.$$

Let

$$b_n=\frac{n^2}{n^3+1}\geq 0.$$

To determine whether $\{b_n\}$ is decreasing, consider the related function

$$f(x)=\frac{x^2}{x^3+1}.$$

Its derivative is

$$f'(x)=\frac{2x(x^3+1)-3x^4}{(x^3+1)^2}=\frac{x(2-x^3)}{(x^3+1)^2}.$$

Thus, $f'(x)<0$ for $x>\sqrt[3]{2}$, so f is decreasing on $[\sqrt[3]{2},+\infty[$. Therefore, $\{b_n\}$ is decreasing for all sufficiently large n. Moreover,

$$\lim_{n\to+\infty}b_n=\lim_{n\to+\infty}\frac{n^2}{n^3+1}=\lim_{n\to+\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^3}}=\frac{0}{1+0}=0.$$

Both conditions of the Alternating Series Test are satisfied. Therefore,

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$$

is convergent.

■ Estimating the Sum of an Alternating Series

Suppose that an alternating series is convergent. Let s denote its sum and s_n its nth partial sum. Recall that the remainder

$$R_n=s-s_n$$

is the error made when s_n is used to approximate s.

Theorem. Alternating Series Error Bound

Suppose

$$s = \sum_{n=1}^{+\infty} (-1)^{n-1} b_n,$$

where $b_n \geq 0$, $b_{n+1} \leq b_n$ for all sufficiently large n, and $\lim_{n\to+\infty} b_n = 0$. Then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

In other words, the error made by using the first n terms of a convergent alternating series is at most the absolute value of the first omitted term.

An Error Bound for an Alternating Series

Consider the series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots .$$

The series satisfies the conditions of the Alternating Series Test. Consider the partial sum

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = 0.368056.$$

The first omitted term has absolute value

$$b_7 = \frac{1}{7!} = \frac{1}{5040} < 0.0002.$$

Therefore, by the Alternating Series Error Bound,

$$|s - s_6| \leq \frac{1}{5040} < 0.0002.$$

Hence,

$$s \approx 0.368$$

correct to three decimal places.

Absolute Convergence

Definition. Absolutely Convergent Series

A series

$$\sum_{n=1}^{+\infty} a_n$$

is called *absolutely convergent* if the series

$$\sum_{n=1}^{+\infty} |a_n|$$

is convergent.

Consider the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}.$$

We have

$$\sum_{n=1}^{+\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2$, the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}$$

is absolutely convergent.

Definition. Conditionally Convergent Series

A series $\sum_{n=1}^{+\infty} a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

The alternating harmonic series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$$

is convergent by the Alternating Series Test. However,

$$\sum_{n=1}^{+\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{+\infty} \frac{1}{n}$$

is the harmonic series, which is divergent. Therefore, the alternating harmonic series is conditionally convergent.

Theorem. Absolute Convergence Implies Convergence

If a series $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose that $\sum_{n=1}^{+\infty} |a_n|$ is convergent. Since

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

and $\sum_{n=1}^{+\infty} 2|a_n|$ is convergent, the Comparison Test implies that

$$\sum_{n=1}^{+\infty} (a_n + |a_n|)$$

is convergent.

For every $N \geq 1$, we have

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (a_n + |a_n|) - \sum_{n=1}^{N} |a_n|.$$

Since both series on the right-hand side are convergent, we can take the limit as $N \to +\infty$:

$$\begin{aligned} \lim_{N \to +\infty} \sum_{n=1}^{N} a_n &= \lim_{N \to +\infty} \sum_{n=1}^{N} (a_n + |a_n|) - \lim_{N \to +\infty} \sum_{n=1}^{N} |a_n| \\ &= \sum_{n=1}^{+\infty} (a_n + |a_n|) - \sum_{n=1}^{+\infty} |a_n|. \end{aligned}$$

Therefore, the sequence of partial sums $\left\{ \sum_{n=1}^{N} a_n \right\}$ converges, so $\sum_{n=1}^{+\infty} a_n$ is convergent. Moreover,

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} (a_n + |a_n|) - \sum_{n=1}^{+\infty} |a_n|.$$

□

Warning. The converse is false. A convergent series is not necessarily absolutely convergent. For example, the alternating harmonic series $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$ is convergent but not absolutely convergent.

A Series with Positive and Negative Terms

Consider the series

$$\sum_{n=1}^{+\infty} \frac{\cos(n)}{n^2}.$$

The signs of the terms do not alternate regularly, so the Alternating Series Test is not appropriate. Instead, consider the series of absolute values:

$$\sum_{n=1}^{+\infty} \left| \frac{\cos(n)}{n^2} \right| = \sum_{n=1}^{+\infty} \frac{|\cos(n)|}{n^2}.$$

Since $|\cos(n)| \leq 1$, we have

$$0 \leq \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}.$$

The series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent. Therefore, by the Comparison Test,

$$\sum_{n=1}^{+\infty} \frac{|\cos(n)|}{n^2}$$

is convergent. Hence,

$$\sum_{n=1}^{+\infty} \frac{\cos(n)}{n^2}$$

is absolutely convergent and therefore convergent.

The Ratio Test

The Ratio Test is particularly useful for series involving exponentials or factorials.

Theorem. The Ratio Test

Let $\sum_{n=1}^{+\infty} a_n$ be a series with $|a_n| > 0$ and suppose that

$$\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then:

1. If $L < 1$, the series is absolutely convergent.
2. If $L > 1$ or $L = +\infty$, the series is divergent.
3. If $L = 1$, the Ratio Test is inconclusive.

Warning.

1. If $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Ratio Test gives no information. The series may converge or diverge. For example, $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent and $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent, but in both cases $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.
2. If $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ with $L > 1$ or $L = +\infty$, the Ratio Test tells us that the series is divergent, meaning that the sequence of partial sums does not converge to a finite real number. There are then two possibilities: either the sequence of partial sums has no limit, or its limit exists in the

extended sense and is equal to $+\infty$ or $-\infty$. For example, for $\sum_{n=1}^{+\infty} 2^n$, we have $\left|\frac{a_{n+1}}{a_n}\right| = 2 > 1$, and

$$\sum_{n=1}^{N} 2^n = 2^{N+1} - 2,$$

so

$$\lim_{N\to+\infty} \sum_{n=1}^{N} 2^n = +\infty.$$

Thus, this series is divergent in the usual sense, but its partial sums tend to $+\infty$, and we may write $\sum_{n=1}^{+\infty} 2^n = +\infty$.

Using the Ratio Test

Consider the series

$$\sum_{n=1}^{+\infty} (-1)^n \frac{n^3}{3^n}.$$

Let

$$a_n = (-1)^n \frac{n^3}{3^n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \frac{3^n}{(-1)^n n^3}\right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3.$$

Therefore,

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1.$$

By the Ratio Test, the series is absolutely convergent and therefore convergent.

Recall: The Exponential Function

Theorem. A Classical Limit for the Exponential Function

For every $x \in \mathbb{R}$,

$$\lim_{n\to+\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

Proof. If $x = 0$, the result is immediate. Suppose that $x \neq 0$. For n sufficiently large, $1 + \frac{x}{n} > 0$, and

$$\ln\left(\left(1+\frac{x}{n}\right)^n\right) = n \ln\left(1+\frac{x}{n}\right) = x \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}.$$

Since

$$\lim_{u\to 0} \frac{\ln(1+u)}{u} = 1,$$

we obtain

$$\lim_{n\to+\infty} n \ln\left(1+\frac{x}{n}\right) = x.$$

By continuity of the exponential function,

$$\lim_{n\to+\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

□

In particular, taking $x = 1$ gives

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e.$$

A Series Involving Factorials

Consider the series

$$\sum_{n=1}^{+\infty} \frac{n^n}{n!}.$$

Let

$$a_n = \frac{n^n}{n!}.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n.$$

Hence, by the theorem above,

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e > 1.$$

Therefore, by the Ratio Test, $\sum_{n=1}^{+\infty} \frac{n^n}{n!}$ is divergent.

5 Power Series

Power Series

Definition. Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where x is a variable and the numbers c_n are constants called the *coefficients* of the series.

For each fixed value of x, a power series becomes a series of real numbers. It may converge for some values of x and diverge for others. Therefore, a power series defines a function

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

whose domain is the set of all values of x for which the series converges.

For example, the geometric series

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

converges if $|x| < 1$ and diverges if $|x| \geq 1$. From the formula for the sum of a convergent geometric series that we established earlier, we have

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1.$$

Definition. Power Series Centered at a

A power series centered at a is a series of the form

$$\sum_{n=0}^{+\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

Notice that a power series centered at a always converges when $x = a$, since

$$\sum_{n=0}^{+\infty} c_n (a-a)^n = c_0.$$

A Power Series Converging Only at Its Center

Consider the power series

$$\sum_{n=0}^{+\infty} n! x^n.$$

For $x = 0$, the series converges. Suppose now that $x \neq 0$ and let

$$a_n = n! x^n.$$

Using the Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = (n+1)|x|.$$

Therefore,

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty.$$

By the Ratio Test, the series diverges for every $x \neq 0$. Hence, the power series converges only for $x = 0$.

■ Determining Where a Power Series Converges

Consider the power series

$$\sum_{n=1}^{+\infty}\frac{(x-3)^n}{n}.$$

Let

$$a_n = \frac{(x-3)^n}{n}.$$

Using the Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\,\frac{n}{(x-3)^n}\right| = \frac{n}{n+1}|x-3|.$$

Therefore,

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$$

By the Ratio Test, the series is absolutely convergent if

$$|x-3| < 1,$$

that is,

$$2 < x < 4.$$

It diverges if $|x-3| > 1$. The Ratio Test gives no information when $|x-3| = 1$, so we must study the endpoints separately.

1. If $x = 4$, then

   $$\sum_{n=1}^{+\infty}\frac{(4-3)^n}{n} = \sum_{n=1}^{+\infty}\frac{1}{n},$$

   which is the harmonic series and is divergent.

2. If $x = 2$, then

   $$\sum_{n=1}^{+\infty}\frac{(2-3)^n}{n} = \sum_{n=1}^{+\infty}\frac{(-1)^n}{n},$$

   which is convergent by the Alternating Series Test.

Therefore, the interval of convergence is

$$[2, 4[.$$

Warning. When the Ratio Test gives an inequality of the form $|x-a| < R$, the endpoints $x = a - R$ and $x = a + R$ must always be studied separately. The Ratio Test is inconclusive at these points.

■ A Power Series Converging for Every Real Number

Consider the power series

$$\sum_{n=0}^{+\infty}\frac{x^n}{n!}.$$

Let

$$a_n = \frac{x^n}{n!}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1}.$$

Therefore, for every fixed $x \in \mathbb{R}$,

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\frac{|x|}{n+1} = 0 < 1.$$

By the Ratio Test, the series converges absolutely for every $x \in \mathbb{R}$.

Thus, the series defines a function

$$f(x) = \sum_{n=0}^{+\infty}\frac{x^n}{n!}.$$

Its first partial sums are

$$s_0(x) = 1,$$

$$s_1(x) = 1 + x,$$

$$s_2(x) = 1 + x + \frac{x^2}{2},$$

$$s_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$

and

$$s_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

These partial sums are polynomials that approximate the function f increasingly well as the number of terms increases. We will later identify this function with the exponential function e^x.

Radius and Interval of Convergence

Theorem. Power Series Convergence

Consider a power series

$$\sum_{n=0}^{+\infty} c_n(x-a)^n.$$

Exactly one of the following three possibilities occurs:

1. The series converges only when $x = a$.
2. The series converges for every $x \in \mathbb{R}$.
3. There exists a number $R > 0$ such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.

Definition. Radius of Convergence

The number R in the theorem above is called the *radius of convergence* of the power series. By convention, $R = 0$ if the series converges only at its center, and $R = +\infty$ if the series converges for every $x \in \mathbb{R}$.

Definition. Interval of Convergence

The *interval of convergence* of a power series is the set of all real numbers x for which the series converges.

If $0 < R < +\infty$, then

$$|x - a| < R$$

is equivalent to

$$a - R < x < a + R.$$

The series converges absolutely inside this interval and diverges outside it. At the endpoints $x = a - R$ and $x = a + R$, either convergence or divergence may occur. Therefore, the interval of convergence can have any one of the following forms:

$$]a - R, a + R[, \qquad]a - R, a + R], \qquad [a - R, a + R[, \qquad [a - R, a + R].$$

Warning. The radius of convergence determines what happens inside and outside the interval, but it does not determine what happens at the endpoints. Each endpoint must be tested separately.

■ Finding a Radius and Interval of Convergence

Consider the power series

$$\sum_{n=0}^{+\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Let

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = 3|x|\sqrt{\frac{n+1}{n+2}}.$$

Therefore,

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x|.$$

By the Ratio Test, the series converges absolutely if

$$3|x| < 1,$$

that is,

$$|x| < \frac{1}{3}.$$

Thus, the radius of convergence is

$$R = \frac{1}{3}.$$

We now test the endpoints.

1. If $x = -\frac{1}{3}$, then

   $$\sum_{n=0}^{+\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n+1}}.$$

   This is a p-series with $p = \frac{1}{2} \leq 1$, so it is divergent.

2. If $x = \frac{1}{3}$, then

   $$\sum_{n=0}^{+\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

   This series converges by the Alternating Series Test.

Therefore, the interval of convergence is

$$\left]-\frac{1}{3},\frac{1}{3}\right].$$

■ Another Radius and Interval of Convergence

Consider the power series

$$\sum_{n=0}^{+\infty}\frac{n(x+2)^n}{3^{n+1}}.$$

This power series is centered at $a=-2$. Let

$$a_n=\frac{n(x+2)^n}{3^{n+1}}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\frac{3^{n+1}}{n(x+2)^n}\right|=\frac{n+1}{n}\frac{|x+2|}{3}.$$

Therefore,

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{|x+2|}{3}.$$

By the Ratio Test, the series converges absolutely if

$$\frac{|x+2|}{3}<1,$$

that is,

$$|x+2|<3.$$

Equivalently,

$$-5<x<1.$$

Thus, the radius of convergence is

$$R=3.$$

We now test the endpoints.

1. If $x=-5$, then
$$\sum_{n=0}^{+\infty}\frac{n(-3)^n}{3^{n+1}}=\frac{1}{3}\sum_{n=0}^{+\infty}n(-1)^n.$$
Since $\lim_{n\to+\infty}n(-1)^n$ does not exist and, in particular, the terms do not tend to 0, the series diverges by the Test for Divergence.

2. If $x=1$, then
$$\sum_{n=0}^{+\infty}\frac{n3^n}{3^{n+1}}=\frac{1}{3}\sum_{n=0}^{+\infty}n.$$
Since $\lim_{n\to+\infty}n\neq 0$, the series diverges by the Test for Divergence.

Therefore, the interval of convergence is

$$]-5,1[.$$

■ Summary: Finding the Interval of Convergence

To determine the interval of convergence of a power series $\sum_{n=0}^{+\infty}c_n(x-a)^n$, we usually proceed as follows:

1. Apply the Ratio Test to determine the values of x for which the series converges absolutely.
2. Rewrite the resulting inequality in the form $|x - a| < R$. The number R is the radius of convergence.
3. Determine the corresponding open interval $]a - R, a + R[$.
4. Test the two endpoints $x = a - R$ and $x = a + R$ separately using another convergence test.
5. Include an endpoint in the interval of convergence exactly when the corresponding series converges.